Introduction to Differential forms
Spring 2011
Exercise 10 (for Wednesday Apr. 13)

$\star$**1.** Fill in the following details to the proof of exactness of the sequence

$$0 \longrightarrow \Omega^k(U_1 \cup U_2) \xrightarrow{I_k} \Omega^k(U_1) \oplus \Omega^k(U_2) \xrightarrow{J_k} \Omega^k(U_1 \cap U_2) \longrightarrow 0\,.$$

(i) Show that $I_\# = (I_k)$ and $J_\# = (J_k)$ are chain maps.

(ii) Show that, in the proof of surjectivity of J_k, that $\omega_1 = \varphi_2 \omega \in \Omega^k(U_1)$ and $\omega_2 = -\varphi_1 \omega \in \Omega^k(U_2)$ when $\omega \in \Omega^k(U_1 \cap U_2)$.

2. Find three different ways to calculate $H^k(\mathbb{R}^3 \setminus L)$ for $k \geq 0$, where $L = \{(x, 0, 0) \colon x \in \mathbb{R}\}$.

3. Let $S = \{(x, y, 0, \ldots, 0) \in \mathbb{R}^n \colon x^2 + y^2 = 1\}$. Calculate $H^k(\mathbb{R}^n \setminus S)$ for every $k \geq 0$.

$\star$**4.** Let $S = S^1 \times \{0\} \subset \mathbb{R}^3$ and $B^2 = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 < 1\}$. Let $\varphi \colon B^2 \times S^1 \to \mathbb{R}^3$ be an embedding so that $\varphi(S^1) = S$. Let $T = \varphi(B^2 \times S^1)$. Calculate $H^k(T)$ and $H^k(T \setminus S)$ for $k \geq 0$.

$\star$**5.** Given open sets $U \subset V \subset \mathbb{R}^n$ the inclusion $\iota \colon U \to V$ defines a *natural direct homomorphism for compactly supported forms (or push-forward)* $\iota_* \colon \Omega^k_c(U) \to \Omega^k_c(V)$ by

$$(\iota_* \omega)_x = \begin{cases} \omega_x, & x \in U \\ 0, & x \notin U. \end{cases}$$

Use push-forwards to show that there exists an exact sequence

$$0 \longrightarrow \Omega^k_c(U_1 \cap U_2) \xrightarrow{I_k} \Omega^k_c(U_1) \oplus \Omega^k_c(U_2) \xrightarrow{J_k} \Omega^k_c(U_1 \cup U_2) \longrightarrow 0$$

where U_1 and U_2 are open maps in $\mathbb{R}^n$ and $I_\# = (I_k)$ and $J_\# = (J_k)$ are chain maps.

6. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets and $A \subset U$ a closed set. Suppose that $f \colon U \to V$ a continuous map that is C^∞-smooth in a neighborhood W of A, and $\varepsilon \colon U \to (0, \infty)$ a continuous function. Show that that there exist a C^∞-smooth map $g \colon U \to V$ so that $g|A = f|A$ and $|f(x) - g(x)| < \varepsilon(x)$ for every $x \in U$.