Introduction to Differential forms
Spring 2011
Exercise 8 (for Wednesday Mar 30.)

1. Let U be a starshaped domain in $\mathbb{R}^n$ about $x_0 \in U$ (that is, $tx+(1-t)x_0 \in U$ for every $x \in U$ and $t \in [0,1]$). Let $\omega \in \Omega^1(U)$. Show that

$$S_1\omega(x) = \int_{\gamma^x} \omega,$$

where $S_1\colon \Omega^1(U) \to \Omega^0(U)$ is the "chain homotopy operator" in the proof of the Poincaré lemma and $\gamma^x\colon [0,1] \to U$ the path $\gamma^x(t) = t(x - x_0) + x_0$.

$\star$**2.** Let U and V be open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and let f and g be *smoothly homotopic* C^∞-mappings $U \to V$, that is, there exists a C^∞-mapping $H\colon U \times \mathbb{R} \to V$ so that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for every $x \in U$. Show that $f^* = g^*\colon H^k(V) \to H^k(U)$ for every $k \geq 0$. (*Hint:* Modify the proof of Poincaré lemma to obtain a "chain homotopy operator" from $\Omega^k(V)$ to $\Omega^{k-1}(U)$.)

$\star$**3.** Let A^*, B^*, and C^* be chain complexes and let $f^\# = (f_k)\colon A^* \to B^*$ and $g^\# = (g_k)\colon B^* \to C^*$ be chain maps, *and* $A^* \xrightarrow{f^\#} B^* \xrightarrow{g^\#} C^*$ *is exact*. Show that $f^*\colon H^k(A^*) \to H^k(B^*)$ and $g^*\colon H^k(B^*) \to H^k(C^*)$ satisfy $\mathrm{Im} f^* \subset \ker g^*$ for every $k \in \mathbb{Z}$.

$\star$**4.** Suppose U and V are disjoint open sets in $\mathbb{R}^n$. Show that, for every k, $H^k(U \cup V) \cong H^k(U) \oplus H^k(V)$.

5.

(i) Let $f\colon A \to B$ be a linear map between vector spaces. Show that $A \cong \mathrm{Im} f \oplus \ker f$.

(ii) Suppose that $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is an exact sequence of vector spaces. Show that B is isomorphic to $A \oplus C$. In particular, that $\dim B = \dim A + \dim C$ if B is finite dimensional.

6[1]**.** Let

$$\begin{array}{ccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5
\end{array}$$

be a commutative diagaram with exact rows. Show that f_3 is injective if f_1 is surjective and f_2 and f_4 are injective. Show that f_3 is surjective if f_5 is injective and f_2 and f_4 are surjective. (If f_1, f_2, f_4 and f_5 are isomorphisms then f_3 is an isomorphism.)

[1]The five lemma