**Introduction to Differential forms**
**Spring 2011**
**Exercise 7 (for Wednesday Mar 23.)**

$\star$**1.** Given a $k$-chain $\sigma = \sum_i a_i\sigma_i$, we denote $|\sigma| = \bigcup_i \sigma_i(\Delta^k)$.

(i) Show that there exists a 2-chain $\sigma$ so that $|\sigma| = S^2$.

(ii) Show that $\sigma$ in (i) can be taken to be a cycle.

(iii) Show that $\sigma$ in (ii) can be taken to be a boundary of a 3-chain in $\mathbb{R}^3$.

**2.** Show that there exists a cycle $\sigma$ as in Ex 1.(ii) and a 2-form $\omega$ in $\mathbb{R}^3 \setminus \{0\}$ so that

$$\int_\sigma \omega \neq 0.$$

It suffices to find a cycle $\sigma$ so that the integral is defined. (Checking the $C^1$-regularity is not required.)

**3.** Show that the compactly supported 0-de Rham cohomology of $\mathbb{R}^n$ is trivial, i.e. $H^0_c(\mathbb{R}^n) = 0$, for every $n \geq 1$.

$\star$**4.**

(i) Let $U$ and $V$ be open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and $f\colon U \to V$ be a $C^\infty$-map. For $k \geq 0$ we define $f^*\colon H^k(V) \to H^k(U)$ by $f^*[\omega] = [f^*\omega]$. Show that $f^*$ is well-defined. Furthermore, show that $f^* \circ g^* = (g \circ f)^*$ when $g\colon V \to W$ is a $C^\infty$-map and $W \subset \mathbb{R}^\ell$ open.

(ii) Let $U$ be an open set in $\mathbb{R}^n$. Show that the exterior product $\wedge\colon H^k(U) \times H^\ell(U) \to H^{k+\ell}(U)$, $[\omega] \wedge [\tau] = [\omega \wedge \tau]$, is well-defined.

**5.** Let $f\colon U \to V$ be a $C^1$-map and $\sigma$ a $C^1$-smooth $k$-chain in $U$. Show that

$$\int_\sigma f^*\omega = \int_{f_*\sigma} \omega$$

for every continuous $k$-form $\omega$ in $V$.