Introduction to Differential forms
Spring 2011
Exercise 4 (for Wednesday Feb 16.)

$\star$**1.**[1] Let V be an n-dimensional vector space and let $\mathcal{B}(V) = \{(v_1, \ldots, v_n) \in V^n : (v_1, \ldots, v_n)$ is a basis of $V\}$. Given $b \in \mathcal{B}$ and a linear map $f\colon V \to V$, let $A[f; \mathfrak{b}] \in \mathbb{R}^{n\times n}$ the matrix of f in basis $\mathfrak{b}$.

For $(v_1, \ldots, v_n), (w_1, \ldots, w_n) \in \mathcal{B}(V)$, we denote $(v_1, \ldots, v_n) \sim (w_1, \ldots, w_n)$ if the matrix $A[f; (v_1, \ldots, v_n)]$ of the mapping $f\colon V \to V$, $v_i \mapsto w_i$, has a positive determinant.

(i) Show that $\sim$ is an equivalence relation in $\mathcal{B}(V)$ and that there are exactly two equivalence classes.

(ii) Let $O(V)$ denote the equivalence classes of orientations as defined in the lecture notes. Show that there exists a well-defined bijection $\Theta_V \colon \mathcal{B}(V)/\sim \to O(V)$ so that $\Theta_V[(v_1, \ldots, v_n)] = [v_1 \wedge \cdots \wedge v_n]$.

$\star$**2.** Let $X\colon \mathbb{R}^n \setminus \{0\} \to T\mathbb{R}^n$ be the vector field $X(p) = (p, p/|p|)$ and let $\omega \in \Gamma^{n-1}(\mathbb{R}^n \setminus \{0\}, \mathrm{Alt}^k(T\mathbb{R}^n))$ be the $(n-1)$-form $\omega_p(v_1, \ldots, v_{n-1}) = dx_1 \wedge \cdots \wedge dx_n(X(p), v_1, \ldots, v_{n-1})$.

(i) Show that

$$\omega = \sum_{i=1}^{n} (-1)^{i+1} \frac{x_i}{|x|} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

where $dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ is the $n-1$-form $dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n$. Conclude that ω is C^∞-smooth in $\mathbb{R}^n \setminus \{0\}$.

(ii) Let $B^{n-1} = \{p \in \mathbb{R}^{n-1} : |p| \le 1$ be the unit disk. We set $\sigma_+ \colon B^{n-1} \to \mathbb{R}^n$ by $\sigma_+(p) = (p, \sqrt{1-|p|^2})$ and $\sigma_- \colon B^{n-1} \to \mathbb{R}^n$ by $\sigma_-(p) = (p, -\sqrt{1-|p|^2})$. Calculate

$$\sigma_+^* \omega \quad \text{and} \quad \sigma_-^* \omega.$$

[1] Another definition for orientation.

(iii) Calculate

$$\int_{B^{n-1}} \sigma_+^*(\omega) \quad \text{and} \quad \int_{B^{n-1}} \sigma_-^*(\omega).$$

when $n = 3$. Here $B^{n-1} \subset \mathbb{R}^{n-1}$ is oriented with $e_1 \wedge \cdots \wedge e_{n-1}$.

$\star$**3.** Let W be a k-dimensional subspace of n-dimensional inner product space V. Let $W^\perp$ be the subspace orthogonal to W, that is, $W^\perp = \{w \in V : \langle w, w' \rangle = 0 \text{ for all } w \in W\}$. Let ξ_W be an orientation of W and ξ_V an orientation of V. Show that there exists an orientation $\xi_{W^\perp}$ of $W^\perp$ so that $\xi_W \wedge \xi_{W^\perp} = \xi_V$.

4. Let (P, ξ) be an oriented k-dimensional affine subspace $P = W + p$ of $\mathbb{R}^n$ and $\xi = e_1' \wedge \cdots \wedge e_k'$, where $(e_1', \ldots, e_k')$ is an orthonormal basis of W. Let $\omega \in C^1(\Gamma^k(T\mathbb{R}^n))$. Show that there exists a constant $c \neq 0$, depending only on k and n, so that

$$\int_P \omega = c \int_P \underline{\omega}_x\left((x, \xi)\right) \, \mathrm{d}\mathcal{H}^k(x),$$

where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure in $\mathbb{R}^n$.

5. Let $U \subset \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}^m$ be a C^1-smooth mapping. One is tempted to define a *tangent* $\mathrm{Tan}_p(f)$ *for the image of f at $f(p)$* to be the subspace $\mathrm{span}\{Df_p(e_1), \ldots, Df_p(e_n)\}$ of $T_{f(p)}\mathbb{R}^m$.

(i) (Pros) Show that $\dim \mathrm{Tan}_p(f) = n$ if and only if $Df_p(e_1) \wedge \cdots \wedge Df_p(e_n) \in \bigwedge_n T_{f(p)}\mathbb{R}^m$ is non-zero.

(ii) (Cons) Give an example of an injective mapping f so that $\dim \mathrm{Tan}_p(f) < n$ for some points $p \in U$. (*Hint:* Look for an easy example when $n = 2$ and $m = 3$.)

6. Denote $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$.

(i) Show that[2] $\pi_1(S^1) = \mathbb{Z}$.

(ii) Show that $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

(iii) Find two different[3] covering spaces[4] for $S^1 \times S^1$.

[2] "Equal" means "isomorphic". $\pi_1(S^1) = \mathbb{Z}$ means $\pi_1(S^1, p) = \mathbb{Z}$ for every $p \in S^1$.

[3] i.e. non-homeomorphic

[4] A space X is an *covering space* of Y if there exists a covering map $X \to Y$