Introduction to Differential forms
Spring 2011
Exercise 3 (for Wednesday Feb 9.)

$\star$**1.** Let W be an m-dimensional vector space and $f\colon V \to W$ a linear map. Show that the map $f_*\colon \bigwedge_k V \to \bigwedge_k W$,

$$f_*\left(\sum_{I=(i_1,\ldots,i_k)} a_I v_{i_1} \wedge \cdots \wedge v_{i_k}\right) = \sum_{I=(i_1,\ldots,i_k)} a_I f(v_{i_1}) \wedge \cdots \wedge f(v_{i_k}),$$

is well-defined and linear. Show also that $f^*\omega(v_1,\ldots,v_k) = \underline{\omega}(f_*(v_1\wedge\cdots\wedge v_k))$ for $v_1,\ldots,v_k \in V$ and $\omega \in \mathrm{Alt}^k(W)$, where $\underline{\omega}\colon \bigwedge_k W \to \mathbb{R}$ is the linearization of ω.

$\star$**2.** Let V be a finite dimensional vector space and W a subspace. Let $\iota\colon W \to V$ be the inclusion $\iota(v) = v$. Show that $\iota_*\colon \bigwedge_k W \to \bigwedge_k V$ is injective and $\iota^*\colon \mathrm{Alt}^k(V) \to \mathrm{Alt}^k(W)$ surjective for all k.

3. Suppose $f\colon \mathbb{R}^4 \to \mathbb{R}^4$ is a linear map so that $\det f = 1$ and $f^*\colon \mathrm{Alt}^2(\mathbb{R}^4) \to \mathrm{Alt}^2(\mathbb{R}^4)$ is identity. Denote $P_{ij} = \mathrm{span}\{\varepsilon_i, \varepsilon_j\}$ for $i, j \in \{1,\ldots,4\}$, where $(\varepsilon_1,\ldots,\varepsilon_4)$ is the standard basis of $(\mathbb{R}^4)^*$.

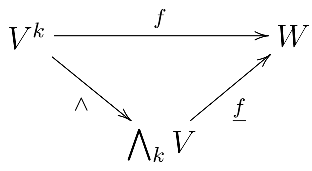
(i) Show that $f^*P_{ij} = P_{ij}$ for all i, j. (*Hint:* For fixed i and j, write $f^*\varepsilon_i = f_i + g_i$, where $f_i \in P_{ij}$, $g_i \in P_{km}$, and $\{i,j,k,m\} = \{1,2,3,4\}$. Argue that $g_i = g_j = 0$ using, e.g., that g_i and g_j are linearly dependent.)

(ii) Show that $f(e_i) = \lambda_i e_i$, where $\lambda_i = \pm 1$, for $i = 1,2,3,4$. (*Hint:* Look what (i) says about f on 2-dimensional planes $\mathrm{span}\{e_i, e_j\}$. Make two such planes intersect. Use geometry!)

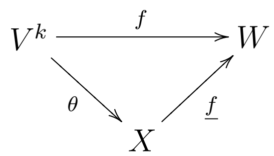
(iii) Find all possible f.

$\star$**4.** Let V and W be finite dimensional vector spaces.

(i) Show that for every alternating multilinear map[1] $f\colon V^k \to W$ there exists a unique linear map $\underline{f}\colon \bigwedge_k V \to W$ so that $\underline{f}(v_1 \wedge \cdots \wedge v_k) = f(v_1, \ldots, v_k)$; as a diagram:



(ii) [2] Suppose X is a vector space with the following property: there exists an alternating multilinear map $\theta\colon V^k \to X$ so that for every finite dimensional space W and every alternating multilinear map $f\colon V^k \to W$ there exists a unique linear map $\underline{f}\colon X \to W$ satisfying



Show that X is isomorphic to $\bigwedge_k V$.

5. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Show that

$$\int_{\mathbb{R}^n} u \circ f |J_f| \, d\mathcal{L}^n = \int_{f(\mathbb{R}^n)} u \, d\mathcal{L}^n$$

where $\mathcal{L}^n$ is the n-dimensional Lebesgue measure and J_f the *Jacobian determinant* $\det f$.

6. (0) Recall from Topology II the definition of a covering map.

(i) Let X and Y be topological spaces, $f\colon X \to Y$ a covering map, and $x_0 \in X$. Let $\sigma\colon [0,1] \to Y$ be a path so that $\sigma(0) = f(x_0)$. Show that there exists a path $\tilde{\sigma}\colon [0,1] \to X$ so that $\tilde{\sigma}(0) = x_0$ and $f \circ \tilde{\sigma} = \sigma$.

(ii) Let $F\colon [0,1]^2 \to Y$ be a map so that $F(0,0) = f(x_0)$. Show that there exists a map $\tilde{F}\colon [0,1]^2 \to X$ so that $\tilde{F}(0,0) = x_0$ and $f \circ \tilde{F} = F$.

(iii) Show that $f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$, $[\gamma] \mapsto [f \circ \gamma]$, is an injection.

[1] A map $f\colon V^k \to W$ is alternating and k-linear if $f(v_1, \ldots, v_k) = 0$ whenever $v_i = v_j$ for $i \neq j$ and $v \mapsto f(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_k)$ is linear for every $v_1, \ldots, v_k \in V$.

[2] The universal property of the exterior product.