Methods II: homotopy invariance

Smooth theory:

Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be open sets, and $f: U \to V$ and $g: U \to V$ C^∞-maps

We say that f and g are smoothly homotopic if exists $H: U \times \mathbb{R} \to V$ C^∞-map so that
$$H(x,0) = f(x) \text{ and } H(x,1) = g(x) \quad \forall x \in U.$$
We denote $f \simeq_\infty g$

We already know: (Ex 8/2)

Lemma: Suppose f and g are smoothly homotopic. Then $f^* = g^*: H^k(V) \to H^k(U)$ $\forall k$

Def. $f: U \to V$ is a (smooth) homotopy equivalence if $\exists$ $g: V \to U$ C^∞-smooth so that $g \circ f \simeq_\infty id_U$ and $f \circ g \simeq_\infty id_V$.

Cor: If f is a smooth h.e then $f^*: H^k(V) \to H^k(U)$ is an isomorphism.

Pf. Let g so that $f \circ g \simeq_\infty id$ and $g \circ f \simeq_\infty id$. Then
$$f^* \circ g^* = (g \circ f)^* = id_U^* = id_{H^k(U)}$$
and
$$g^* \circ f^* = (f \circ g)^* = id_V^* = id_{H^k(V)}$$
for every k. $\square$

Cor: If f is a diffeomorphism (C^∞-homeo with C^∞-inverse) then f^* is an isomorphism.

We will prove:

Thm A: Suppose $A \subset \mathbb{R}^n$ is closed and $A \neq \mathbb{R}^n$.
Then

$$H^{k+1}(\mathbb{R}^{n+1} \setminus (A \times \{0\})) \cong H^k(\mathbb{R}^n \setminus A) \quad \text{for } k \geq 1$$

$$\dim H^1(\mathbb{R}^{n+1} \setminus (A \times \{0\})) = \dim H^0(\mathbb{R}^n \setminus A) - 1$$

$$H^0(\mathbb{R}^{n+1} \setminus (A \times \{0\})) \cong \mathbb{R}$$

We get from Thm A:

Thm: $H^k(\mathbb{R}^n \setminus \{0\}) \cong \begin{cases} \mathbb{R}, & k = 0, n-1 \\ 0, & \text{otherwise} \end{cases}$ $\quad n \geq 2$

Pf: We know that $H^k(\mathbb{R}^2 \setminus \{0\}) \cong \begin{cases} \mathbb{R}, & k = 0, 1 \\ 0, & \text{otherwise} \end{cases}$

We also know that $H^0(\mathbb{R}^n \setminus \{0\}) \cong \mathbb{R}$ for all $n \geq 2$

By Thm A, for $k > 0$,

$$H^k(\mathbb{R}^n \setminus \{0\}) \cong H^{k-1}(\mathbb{R}^{n-1} \setminus \{0\})$$

$$\underset{\text{ind. assumption}}{=} \begin{cases} \mathbb{R}, & k-1 = n-2 \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \mathbb{R}, & k = n-1 \\ 0, & \text{otherwise} \end{cases} \quad \square$$

Cor: $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^m \Rightarrow n = m$

Pf: If $\mathbb{R}^n$ diffeo to $\mathbb{R}^m$, then $\mathbb{R}^n \setminus \{0\}$ diffeo to $\mathbb{R}^m \setminus \{0\}$. Then

$$H^{n-1}(\mathbb{R}^m \setminus \{0\}) \cong H^{n-1}(\mathbb{R}^n \setminus \{0\}) \cong \mathbb{R}$$

Thus $m = n$ $\square$

We postpone the proof of Thm A.

Beyond smooth maps: Continuous maps in de Rham theory

Lemma 1 (Smooth approximation of maps): Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open, $A \subset U$ closed subset. Let $f : U \to V$ and $\varepsilon : U \to (0,\infty)$ be continuous. Then exists C^∞-smooth map $\tilde{f} : U \to V$ so that

$$|f(x) - \tilde{f}(x)| < \varepsilon(x)$$

and, if f is C^∞-smooth in a neighborhood of A,

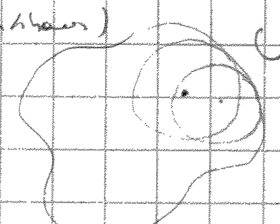
$$f(x) = \tilde{f}(x) \text{ for } x \in A.$$

Pf: Exercise.

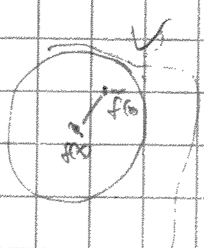
Thm: (1) Every continuous map $f : U \to V$ is homotopic to a C^∞-smooth map $\tilde{f} : U \to V$.

(2) If $f, g : U \to V$ are homotopic smooth maps then they are smoothly homotopic.

Pf: (1) We may assume $V \neq \mathbb{R}^m$.
Let $\varepsilon : U \to (0,\infty)$
$\varepsilon(x) = \operatorname{dist}(f(x), \partial V)$. ($\varepsilon$ continuous)

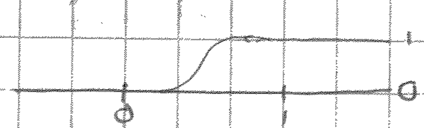


Then $B(f(x), \varepsilon(x)) \subset V$.



Let $\tilde{f} : U \to V$ as in Lemma 1 for this ε.
Then $(1-t) f(x) + t \tilde{f}(x) \in V$ for $0 \le t \le 1$.
Thus $F : U \times [0,1] \to V$, $F(x,t) = (1-t) f(x) + t \tilde{f}(x)$ is a homotopy from f to $\tilde{f}$.

(2) Let $F : U \times [0,1] \to V$ be a homotopy from f to g.
Take continuous $\psi : \mathbb{R} \to [0,1]$ so that



$$\psi(t) = \begin{cases} 1, & t \ge \frac{2}{3} \\ 3(t - \frac{1}{3}), & \frac{1}{3} \le t \le \frac{2}{3} \\ 0, & t < \frac{1}{3} \end{cases}$$

Define $F' : U \times \mathbb{R} \to V$, $F'(x,t) = F(x, \psi(t))$.
Now F' is C^∞-smooth in a neighborhood of $U \times \{0,1\}$.
Apply Lemma 1 as in (1).

<u>Def:</u> Let $f : U \to V$ be continuous. We define

$$f^* : H^k(V) \to H^k(U) \quad \text{by}$$
$$[\omega] \longmapsto [\tilde{f}^*\omega]$$

where $\tilde{f}$ is any smooth map $U \to V$ homotopic to f.

<u>Lemma:</u> f^* is independent on the choice of $\tilde{f}$.

Pf: Suppose $\tilde{f}$ and $\hat{f}$ are smooth maps $U \to V$ homotopic to f. Then $\tilde{f}$ is homotopic to $\hat{f}$.

Then, by Thm, $\tilde{f}$ and $\hat{f}$ are smoothly homotopic. Thus $\tilde{f}^* = \hat{f}^*$. Hence f^* is well-defined. □

The same proof gives: <u>Lemma:</u> If $f, g : U \to V$ are homotopic continuous maps, then $f^* = g^*$.

<u>Cor:</u> If $f : U \to V$ is a homotopy equivalence, then $f^* : H^k(V) \to H^k(U)$ is an isomorphism for every k.

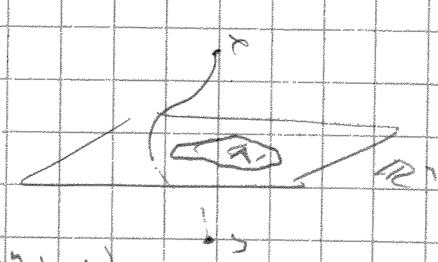
<u>Cor:</u> If U is contractible open set in $\mathbb{R}^n$, then

$$H^k(U) \cong \begin{cases} \mathbb{R}, & k = 0 \\ 0, & k > 0 \end{cases}$$

<u>Pf:</u> Contractible set is connected. Thus $H^0(U) \cong \mathbb{R}$.

Since U is contractible, $id \simeq$ const. map.
Let $f : U \to U$ be a constant map. Then $f^*\omega = 0$ for every $\omega \in \Omega^k(U)$, $k > 0$. Thus $f^* = 0 : H^k(U) \to H^k(U)$.
However, $id^* = f^* = 0$. Thus $H^k(U) = 0$.
□

Pf of Thm A:



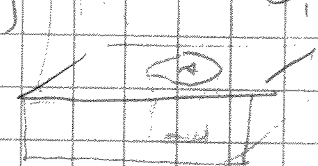
Claim: $H^0(\mathbb{R}^{n+1} \setminus (A \times \{0\})) \cong \mathbb{R}$

Pf: $\mathbb{R}^n \setminus (A \times \{0\})$ is connected.

Claim: $H^{k+1}(\mathbb{R}^{n+1} \setminus (A \times \{0\})) \cong H^k(\mathbb{R}^n \setminus A)$ for $k \geq 1$

Pf: Let $U_1 = \mathbb{R}^n \times (0, \infty) \cup (\mathbb{R}^n \setminus A) \times (-1, \infty)$

$U_2 = \mathbb{R}^n \times (-\infty, 0) \cup (\mathbb{R}^n \setminus A) \times (-\infty, 1)$

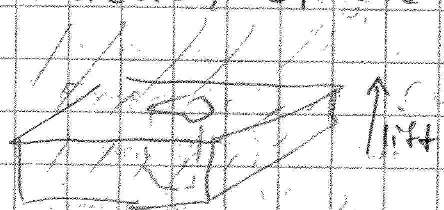


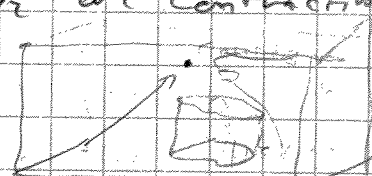
Then

$$U_1 \cup U_2 = \mathbb{R}^{n+1} \setminus (A \times \{0\})$$

and $$U_1 \cap U_2 = (\mathbb{R}^n \setminus A) \times (-1, 1).$$

Moreover, U_1 and U_2 are contractible:





U_1

Contract to e_{n+1} by $(x,t) \mapsto (1-t)x + t e_{n+1}$

By Mayer-Vietoris,

$$\underbrace{H^k(U_1) \oplus H^k(U_2)}_{= 0} \longrightarrow H^k(U_1 \cap U_2) \xrightarrow[\cong]{\partial^*} H^{k+1}(U_1 \cup U_2) \longrightarrow \underbrace{H^{k+1}(U_1) \oplus H^{k+1}(U_2)}_{= 0}$$

for $k \geq 1$

Since $U_1 \cap U_2 = (\mathbb{R}^n \setminus A) \times (-1, 1)$ and
$i : \mathbb{R}^n \setminus A \hookrightarrow (\mathbb{R}^n \setminus A) \times (-1, 1)$ is a smooth homotopy equivalence (Exercise)
we have that

$$H^k(\mathbb{R}^n \setminus A) \cong H^k(U_1 \cap U_2) \cong H^{k+1}(U_1 \cup U_2)$$

<u>Claim:</u> $\dim H^1(\mathbb{R}^n \setminus (A \times \{0\})) = \dim H^0(\mathbb{R}^n \setminus A) - 1$

<u>Pf:</u> By Mayer-Vietoris

$$H^0(U_1) \oplus H^0(U_2) \xrightarrow{J^*} H^0(U_1 \cap U_2) \xrightarrow{\partial^*} H^1(U_1 \cup U_2) \xrightarrow{I^*} \underset{\substack{\| \\ 0}}{H^1(U_1) \oplus H^1(U_2)}$$

Since $\operatorname{Im} I_* = 0$, $\operatorname{Im} \partial^* = \ker I^* = H^1(U_1 \cup U_2)$

Since $U_1 \cap U_2 = (\mathbb{R}^n \setminus A) \times (-1, 1)$, $U_1 \cap U_2$ has as many components as $\mathbb{R}^n \setminus A$. Thus

$$\dim H^0(U_1 \cap U_2) = \dim H^0(\mathbb{R}^n \setminus A)$$

Since $0 \to H^0(U_1 \cup U_2) \xrightarrow{I^*} H^0(U_1) \oplus H^0(U_2) \xrightarrow{J^*} H^0(U_1 \cap U_2)$ is exact and

$$\ker J^* = \operatorname{Im} I^* \cong H^0(U_1 \cup U_2) \cong \mathbb{R}$$

we have
$$\begin{aligned} \dim \operatorname{Im} J^* &= \dim H^0(U_1) \oplus H^0(U_2) - 1 \\ &= 2 - 1 = 1 \end{aligned}$$

Thus $\dim \ker \partial^* = \dim \operatorname{Im} J^* = 1$

Thus
$$\begin{aligned} \dim H^1(U_1 \cup U_2) &= \dim \operatorname{Im} \partial^* = \dim H^0(U_1 \cap U_2) - \dim \ker \partial^* \\ &= \dim H^0(\mathbb{R}^n \setminus A) - 1 \end{aligned}$$

$\square$

<u>Summary:</u> (1) $H^k(\mathbb{R}^n \setminus \{0\}) \cong \begin{cases} \mathbb{R}, & k = 0, n-1 \\ 0, & \text{otherwise} \end{cases}$

(2) $f: U \to V$ homotopy equivalence $\Rightarrow$ $f^*: H^k(V) \to H^k(U)$ isom.

<u>Cor:</u> $\mathbb{R}^n$ homeomorphic to $\mathbb{R}^m$ $\iff$ $n = m$ $\square$

Mappings on the level of cohomology: an example

Problem: Let $A \subset \mathbb{R}^n$ closed and $R : \mathbb{R}^{n+1} \setminus A \times \{0\} \to \mathbb{R}^{n+1} \setminus A \times \{0\}$ be the reflection $R(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n, -x_{n+1})$.

Find mappings $R^\# : H^k(\mathbb{R}^{n+1} \setminus A \times \{0\}) \to H^k(\mathbb{R}^{n+1} \setminus A \times \{0\})$

Ans: Let
$$U_1 = \mathbb{R}^n \times (0, \infty) \cup (\mathbb{R}^n \setminus A) \times (-1, 1)$$
$$U_2 = \mathbb{R}^n \times (-\infty, 0) \cup (\mathbb{R}^n \setminus A) \times (-1, 1)$$
be as in the proof of Thm A.

Step 1: Then
$$\begin{cases} R U_1 = U_2 \\ R U_2 = U_1 \\ R(U_1 \cap U_2) = U_1 \cap U_2 \end{cases}$$
We denote
$$\begin{cases} R_1 = R|U_1 \\ R_2 = R|U_2 \\ R_0 = R|U_1 \cap U_2 \end{cases}$$

Moreover, $R|U_1 \cap U_2 \simeq id_{U_1 \cap U_2}$, since

$$F : U_1 \cap U_2 \times (0, 1) \to U_1 \cap U_2$$
$$((x, t), s) \longmapsto (x, (1-s)(-t) + st)$$

is a homotopy $R|U_1 \cap U_2 \simeq id$.

Furthermore,

$$\begin{array}{ccccc} U_1 \cap U_2 & \xrightarrow{i_1} & U_1 & \xrightarrow{j_1} & U_1 \cup U_2 \\ \downarrow R_0 & & \downarrow R_1 & & \downarrow R \\ U_1 \cap U_2 & \xrightarrow{i_2} & U_2 & \xrightarrow{j_2} & U_1 \cup U_2 \\ \downarrow R_0 & & \downarrow R_2 & & \downarrow R \\ U_1 \cap U_2 & \xrightarrow{i_1} & U_1 & \xrightarrow{j_1} & U_1 \cup U_2 \end{array}$$

commutes, where $i_\ell : U_1 \cap U_2 \to U_\ell$, $j_\ell : U_\ell \to U_1 \cup U_2$ are inclusions as in Thm A.

<u>Step 2:</u> Since $R_0 \simeq id$, we have

$$R_0^* = id : H^k(U_1 \cap U_2) \longrightarrow H^k(U_1 \cap U_2).$$

Moreover, by the proof of Thm A

$$\partial_k^* : H^{k-1}(U_1 \cap U_2) \longrightarrow H^k(U_1 \cup U_2)$$

is surjective for all $k \geq 1$)

(For $k \geq 1$ ∂_k^* is an isom. and $k=0$ just surj.)

So
$$\begin{array}{ccc} H^{k-1}(U_1 \cap U_2) & \xrightarrow[\text{surj.}]{\partial_{k-1}^*} & H^k(U_1 \cup U_2) \\ \downarrow R_0^* = id & & \downarrow R^* \\ H^{k-1}(U_1 \cap U_2) & \xrightarrow[\text{surj.}]{\partial_{k-1}^*} & H^k(U_1 \cup U_2) \end{array}$$

but the diagram does not need to commute!

(If it does: surj ∂_{k-1}^* gives

$$R^*[\omega] = R^* \partial_{k-1}^*[\tau] = \partial_{k-1}^* R_0^*[\tau] = \partial_{k-1}^*[\tau] = [\omega]$$

where $[\tau]$ is such that $\partial_{k-1}^*[\tau] = [\omega]$.)

<u>Idea:</u> Since ∂_{k-1}^* is surj, we can try to calculate "R^* in terms of ∂_{k-1}^*", i.e. compare $R^* \circ \partial_{k-1}^*$ and $\partial_{k-1}^* \circ R_0^*$

<u>Step 3:</u> Definition of ∂_{k-1}^*:

$$\begin{array}{ccccccc} & & \Omega^{k-1}(U_1) \oplus \Omega^{k-1}(U_2) & \xrightarrow{J_{k-1}} & \Omega^{k-1}(U_1 \cap U_2) & \longrightarrow & 0 \\ & & \downarrow d \oplus d & & & & \\ 0 \longrightarrow \Omega^k(U_1 \cup U_2) & \xrightarrow{I_k} & \Omega^k(U_1) \oplus \Omega^k(U_2) & & & & \end{array}$$

$$\partial_{k-1}^*[\omega] = \left[I_k^{-1} (d \oplus d) J_{k-1}^{-1} \omega \right]$$

We calculate $\partial^*_{k-1} \circ R_0^*$ and $R^* \circ \partial^*_{k-1}$:

Let $[\omega] \in H^{k-1}(U_1 \cap U_2)$ and $(\omega_1, \omega_2) \in \Omega^{k-1}(U_1) \oplus \Omega^{k-1}(U_2)$ so that $J_{k-1}(\omega_1, \omega_2) = \omega$, and let $\xi \in \Omega^k(U_1 \cup U_2)$ so that

$$I_k(\xi) = (d\omega_1, d\omega_2) = (i_1^* \xi, i_2^* \xi)$$

Then

$$\begin{aligned} R_0^* \omega &= R_0^* (j_1^* \omega_1 - j_2^* \omega_2) \\ &= (j_1 \circ R_0)^* \omega_1 - (j_2 \circ R_0)^* \omega_2 \\ &= (R_2 \circ j_2)^* \omega_1 - (R_1 \circ j_1)^* \omega_2 \\ &= j_2^* (R_2^* \omega_1) - j_1^* (R_1^* \omega_2) \\ &= - (j_1^* R_1^* \omega_2 - j_2^* R_2^* \omega_1) \\ &= - J_k (R_1^* \omega_2, R_2^* \omega_1) \end{aligned}$$

On the other hand,

$$\begin{aligned} I_k \circ R^* \xi &= (i_1^* R^* \xi, i_2^* R^* \xi) \\ &= ((R \circ i_1)^* \xi, (R \circ i_2)^* \xi) \\ &= ((i_2 \circ R_1)^* \xi, (i_1 \circ R_2)^* \xi) \\ &= (R_1^* i_2^* \xi, R_2^* i_1^* \xi), \end{aligned}$$

Thus

$$\begin{aligned} \partial^*_{k-1} R_0^* [\omega] &= [I_k^{-1} (d \oplus d) J_k^{-1} R_0^* \omega] \\ &= - [I_k^{-1} (d \oplus d)(R_1^* \omega_2, R_2^* \omega_1)] \\ &= - [I_k^{-1} (R_1^* d\omega_2, R_2^* d\omega_1)] \\ &= - [I_k^{-1} (R_1^* i_2^* \xi, R_2^* i_1^* \xi)] \\ &= - [I_k^{-1} \circ I_k \circ R^* \xi] = - R^* [\xi] \end{aligned}$$

Hence $\partial^*_{k-1} = - R^* \partial^*_{k-1}$ for $k \geq 1$. Thus

$$R^* = - \mathrm{id} : H^k(\mathbb{R}^{n+1} \setminus A \times \{0\}) \to H^k(\mathbb{R}^{n+1} \setminus A \times \{0\})$$

for $k \geq 1$.

Case $k=0$: exercise! $\square$