Homology vs. cohomology.

We discuss first briefly general terminology related to homology and cohomology.

Def: A sequence $A_\bullet = (A_k, \partial_k)_{k\in\mathbb{Z}}$ of abelian groups (vector spaces) and homomorphisms (linear maps) $\partial_k : A_k \to A_{k-1}$ is a chain complex if $\partial_{k-1} \circ \partial_k = 0$,

$$\begin{tikzcd} A_k \arrow[r, "\partial_k"] \arrow[rr, bend right, "=0"'] & A_{k-1} \arrow[r, "\partial_{k-1}"] & A_{k-2} \end{tikzcd}.$$

Def: The homology $H_*(A_\bullet)$ of $A = (A_k, \partial_k)$ is the sequence $(H_k(A_\bullet))_{k\in\mathbb{Z}}$ of abelian groups (vector spaces)

$$H_k(A_\bullet) = \ker \partial_k / \operatorname{Im} \partial_k .$$

Example: Let X be a top. space. Take $A_k = C_k(X)$ and $\partial_k = \partial$ as defined before. Then

$$H_k(A) = H_k(X) \quad \text{as defined before.}$$

Example: Let $U \subset \mathbb{R}^n$ open. Take $A_k = \Omega^k(U)$ for $k>0$ and $A_{-k} = \{0\}$ for $k<0$. Take $\partial_k = d$ (exterior deriv.)

Note: $\overset{i\in\mathbb{Z}}{A_i} = \Omega^{-i}(U) \xrightarrow{d} \Omega^{-i+1}(U) = A_{i-1}$

Then $H_k(A_\bullet) = H^k_{dR}(U)$ (as defined before)

"So deRham cohomology is the homology of $(\Omega^k(U), d)$."

Excursion dual sequence

Def: A dual sequence $A^*_\bullet$ (with coefficients in ring R) of $A_\bullet = (A_k, \partial_k)_{k\in\mathbb{Z}}$ is $(A^*_k, \partial^*_k)_{k\in\mathbb{Z}}$

where $A^*_k = \mathrm{Hom}_R(A_k, R) = \{R\text{-homomorphisms } A_k \to R\}$

and

$$\partial^*_k : A^*_{k-1} \to A^*_k \quad \text{defined by} \quad \partial^*_k(c) = c\circ\partial_k$$

Note:

$$\begin{tikzcd} A_k \arrow[r, "\partial_k"] \arrow[dr, "c\circ\partial_k"'] & A_{k-1} \arrow[d, "c"] \\ & R \end{tikzcd}$$

Obs: $\partial^*_{k+1}\circ\partial^*_k = 0$, since $\partial^*_{k+1}\circ\partial^*_k(c) = c\circ\partial_k\circ\partial_{k+1} = 0$.

Def: The cohomology $H^*(A_\bullet;R)$ with coefficients in R is

$$H^k(A_\bullet;R) = \frac{\ker\{\partial^*_{k+1} : A^*_k \to A^*_{k+1}\}}{\ker\{\partial^*_k : A^*_{k-1} \to A^*_k\}}$$

Obs: So "cohomology of $A_\bullet$ is the homology of $A^*_\bullet$."

Arrows change direction: $A_{k-1} \to A_k \to A_{k+1}$

$$A^*_{k-1} \leftarrow A^*_k \leftarrow A^*_{k+1}$$

More examples:

C^∞-chains: Let $U \subset \mathbb{R}^n$ open. Then $\partial : C^\infty C_k(U) \to C^\infty C_{k-1}(U)$ satisfies $\partial^2 = 0$. Thus we have

$$H_{k,\infty}(U) = \frac{\ker\{\partial : C^\infty C_k(U) \to C^\infty C_{k-1}(U)\}}{\mathrm{Im}\{\partial : C^\infty C_{k+1}(U) \to C^\infty C_k(U)\}}$$

is the smooth homology of U.

Rmk 1: Stokes' theorem $\Rightarrow$ $H^k_{dR}(U) \to H^*_{k,\infty}(U)$

This map is actually an isomorphism. (Possibly later)

Rmk 2: Approximation gives $H_k(U) \cong H_{k,\infty}(U)$ (Possibly later)

Compactly supported de Rham:

Let $\Omega^k_c(U) = \{\omega \in \Omega^k(U) : \operatorname{spt}\omega \text{ is compact}\}$

Define $d : \Omega^k_c(U) \to \Omega^{k+1}_c(U)$ to be the exterior derivative.
Note that $\operatorname{spt} d\omega \subset \operatorname{spt}\omega$: $\omega = \sum_I f_I\, dx_I$
If $x \notin \operatorname{spt}\omega$, then $x \notin \operatorname{spt} f_I$ for any I.
Then $x \notin \operatorname{spt} df_I$. Hence $x \notin \operatorname{spt} d\omega$.

Thus d is well-defined. We define

$$H^k_c(U) = \frac{\ker\{d : \Omega^k_c(U) \to \Omega^{k+1}_c(U)\}}{\operatorname{Im}\{d : \Omega^{k-1}_c(U) \to \Omega^k_c(U)\}}.$$

<u>Rmk:</u> We already proved that given $\omega \in \Omega^n_c(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \omega = 0 \quad \text{if and only if} \quad \exists\ \tau \in \Omega^{n-1}_c(\mathbb{R}^n) \text{ s.t. } \omega = d\tau$$

We define now $I : H^n_c(\mathbb{R}^n) \to \mathbb{R}$ by $[\omega] \mapsto \int_{\mathbb{R}^n} \omega$.

This is well-defined: if $[\omega] = [\omega']$, then exists $\tau \in \Omega^{n-1}_c(\mathbb{R}^n)$ so that $\omega - \omega' = d\tau$.
Thus

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} \omega' + \int_{\mathbb{R}^n} d\tau = \int_{\mathbb{R}^n} \omega'.$$

Moreover, I is injective: if $\int_{\mathbb{R}^n} \omega = 0$ then, by thm $\omega = d\tau$ for some $\tau \in \Omega^{n-1}_c(\mathbb{R}^n)$. Thus $[\omega] = 0$.
<u>Consequence:</u> $H^n_c(\mathbb{R}^n) \cong \mathbb{R}$.

The same proof gave also that every $\omega \in \Omega^n(\mathbb{R}^n)$ is of the form $\omega = d\tau$ where $\tau \in \Omega^{n-1}(\mathbb{R}^n)$.
Thus $H^n(\mathbb{R}^n) = 0$.

Fundamental question: How to calculate a homology / cohomology of a given space?

De Rham I

Let $U \subset \mathbb{R}^n$ open.
We set $\Omega^k(U) = \{0\}$ for $k<0$ and $d: \Omega^k(U) \to \Omega^{k+1}$ to be the zero map for $k<0$.

Almost by definition, we have:

Lemma: Let $U \subset \mathbb{R}^n$ connected and open. Then
$$H^0_{dR}(U) \cong \mathbb{R}$$

Pf: By definition
$$H^0_{dR}(U) = \frac{\ker(d: \Omega^0(U) \to \Omega^1(U))}{\operatorname{Im}(d: \Omega^{-1}(U) \to \Omega^0(U))}$$

Since $d: \Omega^{-1}(U) \to \Omega^0(U)$ is the zero map,
$$\ker(d: \Omega^0(U) \to \Omega^1(U)) \Big/ \operatorname{Im} d \cong \ker(d: \Omega^0(U) \to \Omega^1(U))$$
as vector spaces.

On the other hand, $f \in \Omega^0(U)$ satisfies $df = 0$ iff and only if f is locally constant.
Since Ω is connected,
$$\ker(d: \Omega^0(U) \to \Omega^1(U)) = \{f : U \to \mathbb{R} \mid f \text{ const}\} \cong \mathbb{R}$$
as vector spaces

$\square$

The proof gives more:

Lemma: Let $U \subset \mathbb{R}^n$ open. Then
$$H^0_{dR}(U) \cong \mathbb{R}^{\#\{V : V \subset U \text{ components}\}}.$$
(verification as an exercise)

A fundamental fact in de Rham theory is that $H^k(B^n) = \{0\}$ for $k>0$ where $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$. This is usually stated as follows:

Poincaré lemma: Let $U \subset \mathbb{R}^n$ be open and star shaped
(i.e. $\exists\, x_0 \in U$ s.t. $\forall x \in U$ $tx_0 + (1-t)x \in U$ for $0 \le t \le 1$)
Then $H^k(U) = \{0\}$ for $k>0$.

Pf: We need to show that

$$\frac{\ker\left(d: \Omega^k(U) \to \Omega^{k+1}(U)\right)}{\operatorname{Im}\left(d: \Omega^{k-1}(U) \to \Omega^k(U)\right)} = \{0\}$$

for $k > 0$, i.e. for every closed $\omega \in \Omega^k(U)$ is exact.

Idea: for $k > 0$ we find a linear operator

$$S_k: \Omega^k(U) \to \Omega^{k-1}(U)$$

so that

$$\omega = d S_k \omega + S_{k+1} d\omega$$

for every $\omega \in \Omega^k(U)$ i.e. $\quad id = d S_k + S_{k+1} d$

Then, for closed ω, we have

$$\omega = d S_k \omega + S_{k+1} \underbrace{d\omega}_{=0} = d S_k \omega$$

i.e. ω is exact. (Operators S_k are called "chain homotopies")

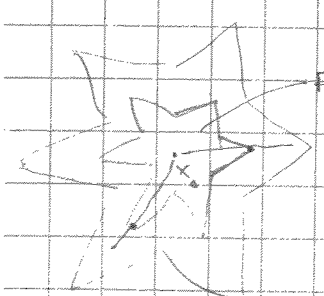
Q: How to find S_k?

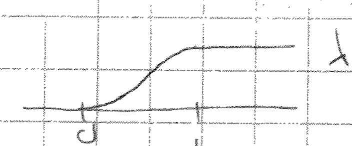
Since U is starshaped, there exists a smooth homotopy $F: U \times \mathbb{R} \to U$ s.t.

$$F(x,t) = x_0 + \lambda(t)(x - x_0)$$

where $\lambda \in C^\infty(\mathbb{R})$ s.t.

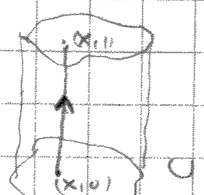
$$\lambda(t) = 0 \quad \text{for } t < 0$$
$$0 \le \lambda(t) \le 1 \quad \text{for } 0 \le t < 1$$
$$\lambda(t) = 1 \quad \text{for } t \ge 1$$





Then $F^*\omega \in \Omega^k(U \times \mathbb{R})$ for every $\omega \in \Omega^k(U)$

Define $\hat{S}_k: \Omega^k(U \times \mathbb{R}) \to \Omega^{k-1}(U)$ by



$$\hat{S}_k\Big(\sum_{\substack{I=(i_1 \dots i_k)\\ 1 \le i_1 < \dots < i_k \le n}} f_I \, dx_I + \sum_{\substack{J=(j_1 \dots j_{k-1})\\ 1 \le j_1 < \dots < j_{k-1} \le n}} g_J \, dt \wedge dx_J\Big)_{(p)}^{\in U}$$

$$= \sum_J \left(\int_0^1 g_J(p,t)\, dt\right)(dx_J)_p = \sum_J G_J \, dx_J$$

We set $S_k = \hat{S}_k \circ F^* : \Omega^k(U) \to \Omega^{k-1}(U)$.

Since
$$d\circ S_k + S_{k+1}\circ d = d\circ\hat{S}_k\circ F^* + \hat{S}_{k+1}\circ F^*\circ d$$
$$= d\circ\hat{S}_k\circ F^* + \hat{S}_{k+1}\circ d\circ F^*$$
$$= (d\circ\hat{S}_k + \hat{S}_{k+1}\circ d)\circ F^*$$
we calculate $d\circ\hat{S}_k$ and $\hat{S}_{k+1}\circ d$.

① $$d\,\hat{S}_k\Big(\sum_I f_I\,dx_I + \sum_J g_J\,dt\wedge dx_J\Big) = d\Big(\sum G_J\,dx_J\Big)$$
$$= \sum dG_J\wedge dx_J$$

where $dG_J = \frac{\partial G_J}{\partial x_1}dx_1 + \dots + \frac{\partial G_J}{\partial x_n}dx_n$ and $\frac{\partial G_J}{\partial x_\ell}(x) = \int_0^1 \frac{\partial g_J}{\partial x_\ell}(x,t)\,dt$

Thus $$(dG_J\wedge dx_J)(x) = \sum_{J,\ell}\Big(\int_0^1 \frac{\partial g_J}{\partial x_\ell}(x,t)\,dt\Big)dx_\ell\wedge dx_J$$

② Similarly,
$$\hat{S}_{k+1}\,d\Big(\overbrace{\sum_I f_I\,dx_I + \sum_J g_J\,dt\wedge dx_J}^{=\tau}\Big)$$
$$= \hat{S}_{k+1}\Big(\sum_I df_I\wedge dx_I + \sum_J dg_J\wedge dt\wedge dx_J\Big)$$
$$= \hat{S}_{k+1}\Big(\sum_I \frac{\partial f_I}{\partial t}dt\wedge dx_I + \underbrace{\sum_{I,i}\frac{\partial f_I}{\partial x_i}dx_i\wedge dx_I}_{\text{does not contain } dt} + \sum_{J,j}\frac{\partial g_J}{\partial x_j}dx_j\wedge dt\wedge dx_J\Big)$$
$$= \hat{S}_{k+1}\Big(\sum_I \frac{\partial f_I}{\partial t}dt\wedge dx_I - \sum_{J,j}\frac{\partial g_J}{\partial x_j}dt\wedge dx_j\wedge dx_J\Big)$$

So
$$(\hat{S}_{k+1}d\tau)_x = \sum_I\Big(\int_0^1\frac{\partial f_I}{\partial t}(x,t)\,dt\Big)dx_I - \sum_{J,j}\Big(\int_0^1\frac{\partial g_J}{\partial x_j}(x,t)\,dt\Big)dx_j\wedge dx_J$$

$$= \sum_I\big(f_I(x,1) - f_I(x,0)\big)dx_I - (dG_J\wedge dx_J)(x)$$

Thus $$(d\hat{S}_k\tau + \hat{S}_{k+1}d\tau)_x = \sum_I\big(f_I(x,1) - f_I(x,0)\big)dx_I$$

for $\tau = \sum_I f_I\,dx_I + \sum_J g_J\,dt\wedge dx_J$

Finally, let $\omega = \sum_I u_I \, dx_I \in \Omega^k(U)$.
Since $F(x,t) = x_0 + \lambda(t)(x - x_0)$, we have

$$F^*(dx_i) = dF_i = d\left(\lambda(t) x_i\right) = \lambda(t)\, dx_i + \lambda'(t) x_i \, dt.$$

Thus $$F^*\omega_{(x,t)} = \sum_I (u_I \circ F)(x,t) \left(\lambda(t)\, dx_{i_1} + \lambda'(t) x_{i_1} dt\right) \wedge \cdots \wedge \left(\lambda(t) dx_{i_k} + \lambda'(t) x_{i_k} dt\right)$$

and $$F^*\omega_{(x,1)} = \sum_I u_I(x) \, dx_{i_1} \wedge \cdots \wedge dx_{i_k} = \omega$$

$$F^*\omega_{(x,0)} = \sum_I u_I(x_0) \cdot 0 = 0$$

Thus $$d S_k \omega + S_{k+1} d\omega = \left(d \hat{S}_k + \hat{S}_{k+1} d\right)(F^*\omega)$$

$$= F^*\omega_{(\cdot,1)} - F^*\omega_{(\cdot,0)}$$

$$= \omega - 0 = \omega. \qquad \square$$

The same argument gives "the homotopy invariance" of de Rham cohomology.

<u>Thm:</u> Let U and V be open in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and f and g be C^∞-smooth maps $U \to V$ so that there exists a C^∞-smooth homotopy $F : U \times \mathbb{R} \to V$ s.t. $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in U$.
Then
$$f^* = g^* : H^k_{dR}(V) \to H^k_{dR}(U)$$
for all $k \geq 0$.

<u>Rmk</u> Note that $f^* : H^k_{dR}(V) \to H^k_{dR}(U)$, $[\omega] \longmapsto [f^*\omega]$, is well-defined by the exercise Ex 7 / Prob. 4.

Pf. of Thm: Exercise

Methods and applications: what is coming next

Given $f: U \to V$ C^∞-map between open sets U and V in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, we obtain a linear map

$$f^*: H^k(V) \to H^k(U) \quad \forall k.$$

Cor: If f is a diffeomorphism, i.e. f^{-1} exists and is C^∞-map, then

$$f^*: H^k(V) \to H^k(U)$$

is an isomorphism for every k.

Pf:
$$(f^{-1})^* \circ f^*[\omega] = (f^{-1})^*[f^*\omega] = [(f^{-1})^*(f^*\omega)]$$
$$= [(f \circ f^{-1})^*\omega] = [\omega]$$
for every $[\omega] \in H^k(V)$

and

$$f^* \circ (f^{-1})^*[\tau] = [(f^{-1} \circ f)^*\tau] = [\tau] \quad \forall [\tau] \in H^k(U)$$

$\square$

To do next:

(1) Mayer-Victoris sequence

$\Rightarrow$ Thm: $H^{n-1}(\mathbb{R}^{n-1} \setminus \{0\}) \neq H^{n-1}(\mathbb{R}^{m-1} \setminus \{0\})$ if $n \neq m$.

$\Rightarrow$ Cor: $\mathbb{R}^n$ is not diffeomorphic to $\mathbb{R}^m$ if $n \neq m$.

(2) Homotopy invariance of de Rham cohomology beyond C^∞-maps.

$\Rightarrow$ Cor: $\mathbb{R}^n \underset{\text{homeo}}{\approx} \mathbb{R}^m \iff n = m$.

$\Rightarrow$ Classical theorems: Brouwer fixed point thm, Invariance of domain, Jordan-Brouwer separation Thm