Manifolds

Basic definitions (see e.g. Holopainen: Riemannian geometry, for more deta

Def: A top. space M is an n-manifold if

- M is locally homeomorphic to $\mathbb{R}^n$ (i.e. $\forall p \in M$ $\exists$ neighborhood U s.t. U is homeo to an open set in $\mathbb{R}^n$)
- M is Hausdorff
- M has countable basis (i.e. is N_2)

Def: $\dim M = n$.

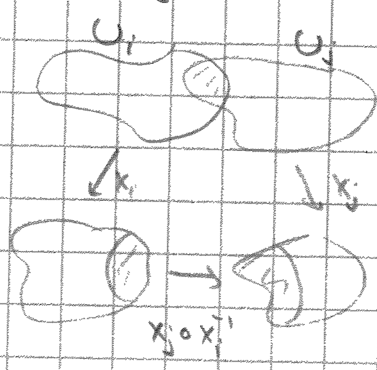
Rmk: dimension is well de by Brouwer's thm

Def: A chart (U, x) on M consists of an open set $U \subset M$ and an embedding $x : U \to \mathbb{R}^n$

Def: A collection $\mathcal{A} = \{(U_i, x_i) : i \in I\}$ is a <u>(smooth) atlas</u> if

(1) $\bigcup_{i \in I} U_i = M$ and

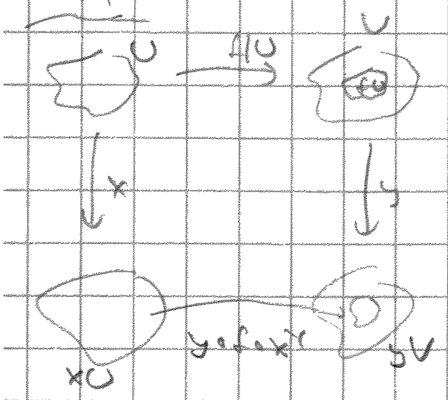
(2) if $U_i \cap U_j \neq \emptyset$, then $x_j^{-1} \circ x_i \mid x_i(U_i \cap U_j)$ is C^∞-map.



Def: A manifold M with a smooth atlas $\mathcal{A}$ is a <u>smooth</u> manifold $(M, \mathcal{A})$

Rmk: We can extend $\mathcal{A}$ to a maximal atlas $\bar{\mathcal{A}}$. From now on we assume that $\mathcal{A}$ is maximal.

Def: A continuous map $f : M \to N$ between smooth manifolds $(M, \mathcal{A})$ and $(N, \mathcal{B})$ is *smooth* if



$$y \circ f \circ x^{-1}|xU : xU \to yV$$

is C^∞-map whenever defined.

Rmks:
- $f : M \to N$ and $g : N \to P$ smooth, then $g \circ f$ smooth
- (U, x) chart on M then $x : U \to \mathbb{R}^n$ smooth when U considered as smooth manifold with structure $\mathcal{A}|U = \{(V, y) \in \mathcal{A} : V \subset U\}$
- identity and inclusion always smooth.

Tangent bundle Suppose M is a smooth manifold.

Let $C^\infty_p(M) = \{ f : U \to \mathbb{R} : U \text{ neighborhood of } p,\ f \text{ smooth}\}$

Def: Linear map $v : C^\infty_p(M) \to \mathbb{R}$ is a *tangent vector* of M at $p \in M$ if

(Leibniz rule) $\quad v(fg) = f(p)\,v(g) + g(p)\,v(f) \quad \forall f, g \in C^\infty_p(M)$

Def: Tangent space of M at $p \in M$ is the vector space

$$T_pM = \{ v : C^\infty_p(M) \to \mathbb{R} : v \text{ is a tangent vector of } M \text{ at } p \in M\}$$

Rmks:
- $\dim T_pM = \dim M = n$; $T_pU = T_pM$ if $p \in U \subset M$ open
- Let (U, x) be a chart and $p \in U$. Then

$$\left(\left(\frac{\partial}{\partial x_1}\right)_p, \ldots, \left(\frac{\partial}{\partial x_n}\right)_p \right) \text{ is a basis of } T_pM$$

where $\left(\frac{\partial}{\partial x_i}\right)_p f = D_{e_i}\left(f \circ x^{-1}\right)(x(p)) = \lim_{h \to 0} \frac{f(x^{-1}(x(p) + he_i)) - f(p)}{h}$

Def: $TM = \bigcup_{p\in M} T_pM$ is the tangent bundle of M.

Let $U \subset M$ open. We denote $TU = \bigcup_{p\in M} T_pM$. =

Rmk: Let (U, x) be a chart of M. Then

$$Dx : TU \longrightarrow xU \times \mathbb{R}^n$$
$$v \longmapsto (p, vx_1, \dots, vx_n)$$

is a bij. so that $Tx \,|\, T_pM : T_pM \longrightarrow \{x(p)\} \times \mathbb{R}^n$ is a linear isom.

Thm: TM is a smooth $2n$-manifold when the topology is defined by $W \subset TM$ open $\iff$ $Dx(W \cap TU)$ open for every $(U,x) \in \mathcal{A}$

and the smooth atlas is given by
$$\{(TU, Tx) : (U,x) \in \mathcal{A}\}.$$

Pf: Exercise. $\square$

Def: A smooth vectorfield is a smooth map $v : M \to TM$ so that $v(p) \in T_pM$ $\forall p \in M$.

Def: Given $f : M \to N$ smooth, we define $Df : TM \to TN$ by
$$(Df)(v)(u) = v(u\circ f)$$
$\forall v \in T_pM$ $\forall u \in C^\infty_{f(p)}(N)$

Forms on manifolds

Def: $Alt^k(TM) = \bigcup_{p\in M} Alt^k(T_pM)$; $Alt^k(TU) = \bigcup_{p\in U} Alt^k(T_pM)$

Rmk: Suppose $f : M \to N$ a smooth map. Let $p \in M$. Then $(Df)_p : T_pM \to T_{f(p)}N$ and, for every $\omega \in Alt^k(T_{f(p)}N)$, we have that $f^*\omega \in Alt^k(T_pM)$ where

$$f^*\omega(v_1 \dots v_k) = \omega((Df)_p v_1, \dots (Df)_p v_k).$$

Def: We say $Alt^k(TM)$ is the <u>k^{th} exterior bundle</u> and say that a section $\omega : M \to Alt^k(TM)$ (that is $\omega(p) \in Alt^k(T_pM)$ $\forall p \in M$) is a <u>k-form</u>. We denote by $\Gamma(M, Alt^k(TM))$ the vector space of sections on M.

Def: We say that $\omega \in \Gamma(M, Alt^k(TM))$ is cont./smooth/... if for every chart (U, x), the form $(x^{-1})^*\omega \in \Gamma(xU, T\mathbb{R}^n)$ is continuous/smooth/...

Fact: There exists a topology on $Alt^k(TM)$ making it an $(n+\binom{n}{k})$-manifold so that

$$\begin{aligned} (x^{-1})^* : Alt^k(TU) &\longrightarrow (xU) \times \underbrace{Alt^k(\mathbb{R}^n)}_{\cong \mathbb{R}^{\binom{n}{k}}} \\ \omega_p &\longmapsto (x(p), (x^{-1})^*\omega_p) \end{aligned}$$

is a homeomorphism (Here we have identified $Alt^k(T_{x(p)}\mathbb{R}^n)$ with $Alt^k(\mathbb{R}^n)$)

Moreover, $\{(Alt^k(TU), (x^{-1})^*) : (U, x) \in \mathcal{A}\}$ is a smooth atlas on $Alt^k(TM)$.

(Proofs as in the case of the tangent bundle)

Def: We denote by $\Omega^k(M)$ the smooth k-forms on M and by $\Omega^k_c(M)$ compactly supported smooth k-forms on M.

Exterior derivative

Let M be a smooth manifold and $\omega \in \Omega^k(M)$.

Def: A form $\tau \in \Omega^{k+1}(M)$ is an exterior derivative of ω if

$$(1) \qquad d(x^{-1})^*\omega = (x^{-1})^*\tau$$

for every chart (U,x) on M. We denote $d\omega = \tau$.

Rmk: (Uniqueness) Suppose $\omega \in \Omega^k(M)$ and $\tau, \tau' \in \Omega^{k+1}(M)$ are such that (1) holds. Let $p \in M$ and (U,x) be a chart at p. Then

$$\tau_p = x^*((x^{-1})^*\tau)_{x(p)} = x^*(d(x^{-1})^*\omega)$$

$$= x^*((x^{-1})^*\tau')_{x(p)} = \tau'_p.$$

Thus $\tau = \tau'$. $\square$

Rmk: (Existence) Suppose $\omega \in \Omega^k(M)$. Let $p \in M$ and (U,x) and (V,y) be charts so that $p \in U \cap V$. Then, on $U \cap V$,

$$x^* d(x^{-1})^*\omega = (x \circ y^{-1} \circ y)^* d(y^{-1} \circ y \circ x^{-1})^*\omega$$

$$= y^*(x \circ y^{-1})^* d(y \circ x^{-1})^*(y^{-1})^*\omega$$

$$= y^*(x \circ y^{-1})^*(y \circ x^{-1})^* d(y^{-1})^*\omega$$

$$= y^* d(y^{-1})^*\omega.$$

Define $\tau : M \to \mathrm{Alt}^k(TM)$ by $\tau|U = x^* d(x^{-1})^*\omega$ on chart (U,x). Then τ is well-defined (independent on the choice of the chart.)

Moreover, on (U,x) we have

$$(x^{-1})^*\tau = (x^{-1})^* x^* d(x^{-1})^*\omega = d(x^{-1})^*\omega.$$

Then $\tau = d\omega$. □

Rmk: By construction $(x^{-1})^* d\omega = d(x^{-1})^*\omega$.

Rmk: τ is an exterior derivative of ω if for every $p \in M$ exists a chart (U,x) at p so that $(x^{-1})^*\tau = d(x^{-1})^*\omega$. (2)

Pf: Suppose that τ satisfies this condition and (V,y) is a chart on M. Let $p \in V$ and (U,x) at p satisfy (2). Then

$$(y^{-1})^*\tau = (x\circ y^{-1})^*(x^{-1})^*\tau = (x\circ y^{-1})^* d(x^{-1})^*\omega$$
$$= d(x\circ y^{-1})^*(x^{-1})^*\omega = d(y^{-1})^*\omega$$

on $y(U\cap V)$. Thus $(y^{-1})^*\tau = d(y^{-1})^*\omega$ on $y(U)$. □

Lemma: (1) The map $d: \Omega^k(M) \to \Omega^{k+1}(M)$ is well-defined and linear.

(2) $d(\omega\wedge\tau) = d\omega\wedge\tau + (-1)^k\omega\wedge d\tau$ $\quad\forall \omega\in\Omega^k(M)$, $\tau\in\Omega^l(M)$

(3) $d\circ d = 0$

(4) $d\circ f^* = f^*\circ d$ for all smooth maps $f: M\to N$.

Pf: (1) holds by remarks; we omit (2) and (3).

(4): Let $\omega\in\Omega^k(M)$ and $p\in M$. Let (U,x) be a chart on M at p s.t. $fU\subset V$, where (V,y) is a chart on N. Then

$$d((x^{-1})^* f^*\omega) = d((x^{-1})^*(y\circ f)^*(y^{-1})^*\omega)$$
$$= d((y\circ f\circ x^{-1})^*(y^{-1})^*\omega) = (y\circ f\circ x^{-1})^* d(y^{-1})^*\omega$$
$$= (x^{-1})^* f^* y^* d(y^{-1})^*\omega = (x^{-1})^* f^* y^*(y^{-1})^* d\omega$$
$$= (x^{-1})^* f^* d\omega$$

□

We conclude from the lemma that

$$\to \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \Omega^{k+2}(M) \to \cdots$$

is a chain complex and

$$\begin{array}{ccccccc} \to & \Omega^k(N) & \xrightarrow{d} & \Omega^{k+1}(N) & \longrightarrow & \Omega^{k+2}(N) & \to \\ & \downarrow f^* & & \downarrow f^* & & \downarrow f^* & \\ \to & \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) & \longrightarrow & \Omega^{k+2}(M) & \longrightarrow \end{array}$$

commutes. Thus $\left(f^* : \Omega^k(N) \to \Omega^k(M)\right)_{k\in\mathbb{Z}}$ is a chain map.
for every smooth $f: M \to N$.

<u>Def</u>: The k^{th} de Rham cohomology of M is

$$H^k(M) = \frac{\ker\{d : \Omega^k(M) \to \Omega^{k+1}(M)\}}{\operatorname{Im}\{d : \Omega^{k-1}(M) \to \Omega^k(M)\}}.$$

<u>Rmk</u>: Given $f: M \to N$ smooth, then $f^*: H^k(N) \to H^k(M)$,
$[\omega] \mapsto [f^*\omega]$, as in the Euclidean case. □

We need partition of unity to translate proofs from the Euclidean case to the manifold setting.

<u>Thm</u> (Partition of unity)
Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ be an open cover of a smooth manifold M.
Then there exist smooth functions $\psi_k : M \to [0,1]$ with compact support so that

(1) $\forall k \ \exists \alpha$ s.t. $\operatorname{spt}\psi_k \subset U_\alpha$

(2) $\forall p\in M \ \exists$ a neighborhood W_p s.t. $\#\{k : \operatorname{spt}\psi_k \cap W_p \neq \emptyset\} < \infty$

(3) $\sum_k \psi_k \equiv 1$

Proof is similar to the Euclidean case and is based on the following topological observations:

Lemma 1: The topology of M has a countable basis whose elements have compact closure.

Pf: Let $\mathcal{B}$ be a countable basis of M.
Set $\mathcal{S} = \{ B \in \mathcal{B} : \overline{B} \text{ is compact} \}$.
Then $\mathcal{S}$ is a basis. Indeed, let $U \subset M$ be open and $p \in U$. Let (V, y) be a chart s.t. $p \in V$.
Then $\exists$ a ball B' in $y(V \cap U)$ containing $y(p)$ so that $\overline{B'} \subset y(V \cap U)$. Then $\exists$ $B \in \mathcal{B}$ s.t.
$B \subset y^{-1}(B')$. Since $\overline{B} \subset y^{-1}(\overline{B'})$ is compact
$B \in \mathcal{S}$. Then $p \in B \subset V \cap U \subset U$ □

Lemma 2: There exists open sets $V_1 \subset \overline{V_1} \subset V_2 \subset \ldots$ on M so that $M = \bigcup_k V_k$ and $\overline{V_k}$ is compact.

Pf: Let $\mathcal{B}$ be a countable basis with sets whose closure is compact; $\mathcal{B} = \{ B_1, B_2, \ldots \}$

Let $V_1 = B_1$ and define V_{k+1} to be

$$V_{k+1} = B_1 \cup \ldots \cup B_{m_k}$$

where m_k is minimal number such that

$$\overline{V_k} \subset B_1 \cup \ldots \cup B_{m_k}.$$

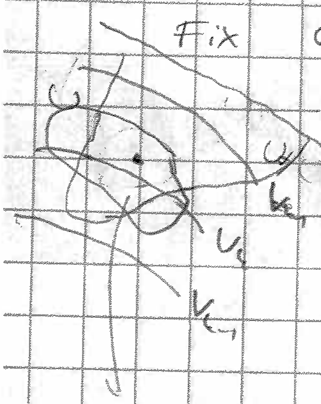
□

Sketch of the proof of Partition of Unity

(for more details see exercises in the Euclidean case)

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ be an open cover and $V_1 \subset \overline{V_1} \subset V_2 \subset \dots$ as in Lemma 2.

Let $p \in M$. Fix $\alpha \in I$ s.t. $p \in U_\alpha$ and let k be so that $p \in V_{k+2} \setminus \overline{V_{k-1}}$. Let (U, x) be chart at p.

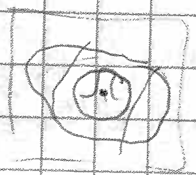


Fix $\varphi_p \in C_0^\infty(x(U \cap U_\alpha \cap (V_{k+2} \setminus \overline{V_{k-1}})))$ non-negative so that $\varphi_p(x(p)) > 0$

Define $\tilde{\varphi}_p = \begin{cases} \varphi_p \circ x & \text{on } U \\ 0, & \text{otherwise} \end{cases}$

Then $\tilde{\varphi}_p \in C^\infty(M)$ s.t. $\tilde{\varphi}_p(p) > 0$.

Let $W_p = \tilde{\varphi}_p^{-1}(0, \infty)$. Then $W_p \subset V_{k+2} \setminus \overline{V_{k-1}}$



Since $\{W_p : p \in V_{k+2} \setminus \overline{V_{k-1}}\}$ is an open cover of $\overline{V_{k+1}} \setminus V_k$ which is compact, there exists a finite subcover $\{W_{p_{k,1}}, \dots, W_{p_{k,m_k}}\}$ covering $\overline{V_{k+1}} \setminus V_k$.

We relabel $\{\tilde{\varphi}_{k,i} : k \geq 1, 1 \leq i \leq m_k\} = \{\tilde{\varphi}_1, \tilde{\varphi}_2, \dots\}$

By local finiteness, $\tilde{\varphi} = \sum_i \tilde{\varphi}_i$ is well-defined, smooth and positive.

We set $\varphi_k = \frac{\tilde{\varphi}_k}{\tilde{\varphi}}$ for $k \geq 1$.

These functions satisfy the claim □

Basic de Rham theory on manifolds

Mayer–Vietoris

The proof of Mayer–Vietoris sequence uses only properties of chain complexes and the existence of partition of unity. Thus we have, with the same proof as in the Euclidean case:

Thm: (Mayer–Vietoris)
Let $U_1, U_2 \subset M$ be open sets on a smooth n-manifold M. Then there exists $\partial_k^*: H^k(U_1 \cap U_2) \to H^{k+1}(U_1 \cup U_2)$ linear s.t. the

$$\cdots \to H^k(U_1 \cup U_2) \xrightarrow{I_k^*} H^k(U_1) \oplus H^k(U_2) \xrightarrow{J_k^*} H^k(U_1 \cap U_2) \xrightarrow{\partial_k^*} H^{k+1}(U_1 \cup U_2) \to$$

is exact, where $I_k : \Omega^k(U_1 \cup U_2) \to \Omega^k(U_1) \oplus \Omega^k(U_2)$ is given by $I_k = (i_1^*, i_2^*)$ and $i_\ell : U_1 \cap U_2 \to U_\ell$ incl. and $J_k = j_1^* - j_2^* : \Omega^k(U_1) \oplus \Omega^k(U_2) \to \Omega^k(U_1 \cap U_2)$ where $j_\ell : U_\ell \to U_1 \cup U_2$ are inclusions.

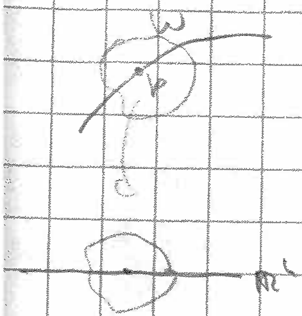
Homotopy invariance

Method I: (Full topological result)

We need two theorems:

Thm: (Whitney) Every smooth manifold can be embedded as a smooth submanifold of a Euclidean space.

Rmk: In fact every top. n-manifold can be embedded into $\mathbb{R}^{2n+1}$.
(Whitney's thm)

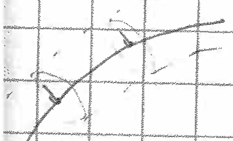


Def: A set $N \subset M$ is a <u>smooth k-submanifold</u> of a smooth manifold M if every $p \in N$ has a neighborhood W so that exists a diffeomorphism $\varphi : W \to W' \subset \mathbb{R}^n$ so that $\varphi(W \cap N) = W' \cap \mathbb{R}^k \times \{0\}$.

Rmk: 1. A smooth submanifold is a smooth manifold. (N locally homeo to $\mathbb{R}^k$, N is Hausdorff and N_2 by inherited relative topology; transition maps are restrictions hence smooth.)

Thm: (Tubular neighborhood Thm.)

Let $N \subset \mathbb{R}^n$ be a smooth submanifold. Then exists a neighborhood W of N in $\mathbb{R}^n$ and a smooth map $\pi : W \to N$ so that



(1) $\pi | N = \mathrm{id}$

(2) for every $y \in W$, $|y - \pi(y)| = \min_{x \in N} |y - x|$

We omit the proofs of these theorems (see [Madsen–Tornehave Thm 9.11 and 9.23].)
These give us the following

Thm (Approximation): Let $f, g : M \to N$ be homotopic continuous maps where M and N are smooth manifolds.

(1) There exists smooth maps $\tilde{f}, \tilde{g} : M \to N$ so that $\tilde{f} \simeq f$ and $\tilde{g} \simeq g$ and $\tilde{f}$ is smoothly homotopic to $\tilde{g}$.

(2) $f^* : H^k(N) \to H^k(M)$, $f^*[\omega] = \tilde{f}^*[\omega]$, where $\tilde{f}$ is any smooth map homotopic to f, is well-defined, and $f^* = g^* : H^k(N) \to H^k(M)$.

(3) If f is smooth in a neighborhood of $A \subset M$ then there exists a smooth map $\tilde{f} : M \to N$ homotopic to f so that $\tilde{f}|A = f|A$.

<u>Proof:</u> Let $\varphi : M \to \mathbb{R}^m$ and $\psi : N \to \mathbb{R}^k$ be smooth embeddings of M and N into Euclidean spaces. By Tubular Neighborhood Thm, there exists $U \supset \varphi(M)$ and $V \supset \psi(N)$ and projections $\pi_M : U \to \varphi M$ and $\pi_N : V \to \psi N$.

Thus $f' : U \to V$, $f' = \psi \circ f \circ \varphi^{-1} \circ \pi_M$, and $g' : U \to V$, $g' = \psi \circ g \circ \varphi^{-1} \circ \pi_M$ are continuous homotopic maps. Indeed, let
$F : M \times [0,1] \to N$ be homotopy from f to g.
Then
$$F' : U \times [0,1] \to V, \quad F'(x,t) = \psi\left(F(\varphi^{-1} \circ \pi_M(x), t)\right),$$
is a homotopy $f' \simeq g'$.

Since $U \subset \mathbb{R}^m$ is open and $V \subset \mathbb{R}^k$ is open, there exists smooth maps $\tilde{f}', \tilde{g}' : U \to V$ so that $\tilde{f}' \simeq f'$, $\tilde{g}' \simeq g'$, and $\tilde{f}' \simeq_\infty \tilde{g}'$.

Thus maps $\tilde{f} : M \to N$, $\tilde{g} : M \to N$,
$$\tilde{f} = \psi^{-1} \circ \pi_N \circ \tilde{f}' \circ \varphi, \quad \tilde{g} = \psi^{-1} \circ \pi_N \circ \tilde{g}' \circ \varphi$$
are smoothly homotopic smooth maps so that
$$\tilde{f} \simeq f \quad \text{and} \quad \tilde{g} \simeq g.$$

This proves (1). The cases (2) & (3) follow as in the Euclidean case. $\square$

<u>Method II</u>

Using the proof of the Poincaré lemma, we get

<u>Thm:</u> (Poincaré lemma) Suppose $f, g : M \to N$ are smoothly homotopic smooth maps. Then exists linear maps $S_k : \Omega^k(N) \to \Omega^{k-1}(M)$ so that
$$f^*\omega - g^*\omega = d S_k \omega + S_{k+1} d\omega$$

<u>Pf:</u> Exercises. $\square$

<u>Cor:</u> For smoothly homotopic smooth maps $f, g : M \to N$ we have

$$f^* = g^* : H^k(N) \to H^k(M).$$

<u>Pf:</u> Let $[\omega] \in H^k(N)$. Then

$$f^*[\omega] = [f^*\omega] = [g^*\omega + dS_t\omega + S_{t+1}\underbrace{d\omega}_{=0}]$$

$$= [g^*\omega] + [dS_t\omega] = [g^*\omega] = g^*[\omega]$$

$\square$

<u>Rmk:</u> The approximation theorem now gives that given a continuous map $f : M \to N$ there exists a smooth map $\tilde{f} : M \to N$ homotopic to f. The approximation theorem can be proved without Whitney embedding. See exercises for the existence of $\tilde{f}$.