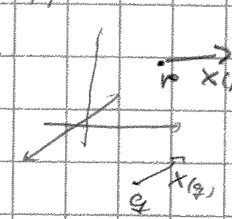
Introduction

Multivariable calculus

Let $X : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a vector field (C^1, say) and let γ be a curve $\gamma : [a,b] \longrightarrow \mathbb{R}^n$, C^1 say,

$\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$, where $\gamma_i : [a,b] \longrightarrow \mathbb{R}$

Then integral of X over γ is



$$\int_a^b \langle X(\gamma(t)), \dot\gamma(t) \rangle \, dt$$

where $\langle \cdot , \cdot \rangle$ is the (dot) inner product and $\dot\gamma(t) = (\gamma_1'(t), \dots, \gamma_n'(t))$

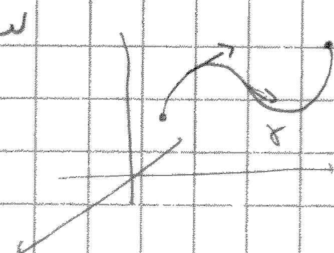
<u>Observation</u>: Set $\omega_p(v) = \langle v, X(p) \rangle$.

Then

$$\int_a^b \langle X(\gamma(t)), \dot\gamma(t) \rangle \, dt = \int_a^b \omega_{\gamma(t)}(\dot\gamma(t)) \, dt$$

$$\text{Set} \quad \int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\dot\gamma(t)) \, dt.$$

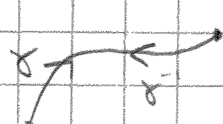
$$\text{So} \quad \int_a^b \langle X(\gamma(t)), \dot\gamma(t) \rangle \, dt = \int_\gamma \omega$$



ω is a called a 1-form

<u>Observation</u>: Set $\overleftarrow{\gamma} : [a,b] \longrightarrow \mathbb{R}^n$

$\overleftarrow{\gamma}(t) = \gamma(b - (t-a)) = \gamma(\alpha(t))$



where

$\alpha : [a,b] \to [a,b]$, $\alpha(t) = b - (t-a)$

$$\text{Then} \quad \int_{\overleftarrow{\gamma}} \omega = - \int_\gamma \omega.$$

Lecture 2/2

Indeed, consider $u(t) = \langle X(\gamma(t)), \dot\gamma(t)\rangle$

Then $\langle X(\overleftarrow{\gamma}(t)), \dot{\overleftarrow{\gamma}}(t)\rangle = \langle X(\gamma\circ\alpha(t)), \dot{(\gamma\circ\alpha)}(t)\rangle$

$$= \langle X(\gamma(\alpha(t))), \dot\gamma(\alpha(t))\underbrace{\alpha'(t)}_{=-1}\rangle$$

$$= -\langle X(\gamma(\alpha(t))), \dot\gamma(\alpha(t))\rangle$$

$$= -u(\alpha(t)).$$

$\overleftarrow{\gamma}(t) = \gamma\circ\alpha(t) = (\gamma_1\circ\alpha(t), \ldots, \gamma_n\circ\alpha(t))$ & $(\gamma_k\circ\alpha)'(t) = \gamma_k'(\alpha(t))\alpha'$

Thus
$$\int_{\overleftarrow{\gamma}} \omega = \int_a^b \langle X(\overleftarrow{\gamma}(t)), \dot{\overleftarrow{\gamma}}(t)\rangle\, dt = -\int_a^b u(\alpha(t))\, dt$$

$$= -\int_{\alpha^{-1}(a)}^{\alpha^{-1}(b)} u(\alpha(\alpha^{-1}(t)))\underbrace{(\alpha^{-1})'(t)}_{=-1}\, dt = \int_b^a u(t)\, dt$$

Change of variables

$$= -\int_a^b \omega_{\gamma(t)}(\dot\gamma(t))\, dt = -\int_\gamma \omega \qquad \square$$

Change of "direction"

Observation 2: If $X = \nabla f$, then
$$\int_a^b \langle X(\gamma(t)), \dot\gamma(t)\rangle dt = f(\gamma(b)) - f(\gamma(a))$$

Short term goals:

1. Understand 1-forms better
2. Generalize 1-forms and integration to surfaces and higher dimensions.
   - "understanding direction" (Alternating algebra)
3. Find "proper" versions of paths (Homology & cohomology)

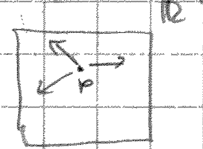
$$\{\text{paths}\} \times \{1\text{-forms}\} \xrightarrow{\text{integration}} \mathbb{R}$$

$$(\gamma, \omega) \longmapsto \int_\gamma \omega$$

Q: What are the ...

Vector fields, forms, and "d" I

Tangent space



We define

$$T_p\mathbb{R}^n = \{p\} \times \mathbb{R}^n \quad \text{for every } p \in \mathbb{R}^n.$$

Def: $T_p\mathbb{R}^n$ is the tangent space of $\mathbb{R}^n$ at $p \in \mathbb{R}^n$.

Observation: $T_p\mathbb{R}^n$ is a vector space (isom to $\mathbb{R}^n$) with operations

$$+ : T_p\mathbb{R}^n \times T_p\mathbb{R}^n \longrightarrow T_p\mathbb{R}^n \quad (p,v) + (p,w) = (p, v+w)$$

$$\cdot : \mathbb{R} \times T_p\mathbb{R}^n \longrightarrow T_p\mathbb{R}^n \quad a(p,v) = (p, av).$$

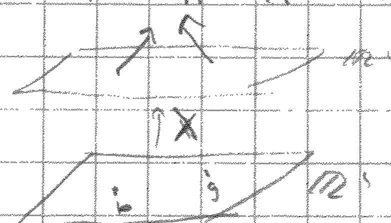
Observation: $\mathbb{R}^n \times \mathbb{R}^n = \bigcup_{p\in\mathbb{R}^n} \{p\} \times \mathbb{R}^n = \bigcup_{p\in\mathbb{R}^n} T_p\mathbb{R}^n$

Def: $T\mathbb{R}^n = \bigcup_{p\in\mathbb{R}^n} \{p\} \times \mathbb{R}^n$ is the tangent bundle of $\mathbb{R}^n$.

Let $\pi : T\mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the projection $(p,v) \longmapsto p$.

Def: Let $E \subset \mathbb{R}^n$ be a set. A map $X : E \longrightarrow T\mathbb{R}^n$ is a vector field if $\pi \circ X = id$.

$$\begin{tikzcd} T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n \arrow[d, shift left, "\pi"] \\ \mathbb{R}^n \arrow[u, shift left, "X"] \arrow[loop left, "id"] \end{tikzcd}$$



<u>Def:</u> Given $\gamma : [a,b] \to \mathbb{R}^n$, we redefine $\dot\gamma : [a,b] \to T\mathbb{R}^n$ by $\dot\gamma(t) = (\gamma(t), \gamma_1'(t), \dots \gamma_n'(t))$.

<u>Cotangent space</u>

Let V, W be vector spaces (fin. dim)

We define $\mathcal{L}(V, W) = \{ f : V \to W : f \text{ linear} \}$

<u>Def:</u> Dual V^* of V is $V^* = \mathcal{L}(V, \mathbb{R})$.

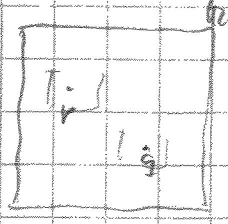
Thm: Suppose $\{e_1, \dots e_n\}$ is a basis of V
(Lin. alg.) Then $\{\varepsilon_1 \dots \varepsilon_n\}$ is a basis of V^*, where $\varepsilon_k : V \to \mathbb{R}$ is the (unique) linear map $\varepsilon_k(e_i) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$

<u>Cor:</u> Every linear map $f : V \to \mathbb{R}$ has a representation
$$f = \sum_{k=1}^n b_k \varepsilon_k \quad \text{where } b_k \in \mathbb{R} \text{ (unique)}.$$

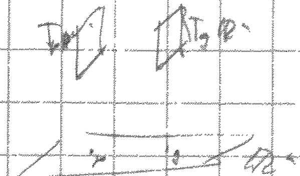
<u>Def:</u> Cotangent space of $\mathbb{R}^n$ at $p \in \mathbb{R}^n$ is

$$T_p^*\mathbb{R}^n = (T_p\mathbb{R}^n)^* \left(= \mathcal{L}(\{p\} \times \mathbb{R}^n, \mathbb{R}) \right).$$

<u>Observation:</u> $f \in T_p^*\mathbb{R}^n \Rightarrow f : T_p\mathbb{R}^n \to \mathbb{R}$ linear map.



If $p \neq q$, then $T_p\mathbb{R}^n \neq T_q\mathbb{R}^n$. Thus $T_q^*\mathbb{R}^n$ and $T_p^*\mathbb{R}^n$ contain different maps.



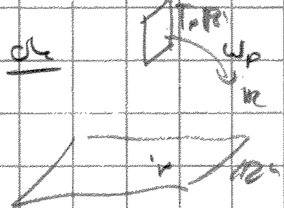
<u>Exercise:</u> Find formulas for isomorphisms
$$(T_p\mathbb{R}^n)^* \cong \{p\} \times (\mathbb{R}^n)^* \cong \mathbb{R}^n$$

<u>Def</u> $T^*\mathbb{R}^n = \bigcup_{p\in\mathbb{R}^n} T_p^*\mathbb{R}^n$ is the cotangent bundle of $\mathbb{R}^n$.

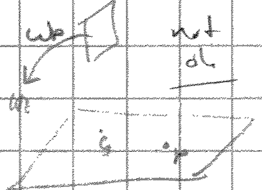
We denote $\pi : T^*\mathbb{R}^n \longrightarrow \mathbb{R}$ $\quad f \longmapsto p \quad$ if $f\in T_p\mathbb{R}^n$.

<u>Exercise:</u> Using the isomorphisms of the previous exercise find bijections
$$T^*\mathbb{R}^n \cong \mathbb{R}^n\times(\mathbb{R}^n)^* \cong \mathbb{R}^{2n}$$

<u>Def:</u> Let $E\subset\mathbb{R}^n$ a set. A map $\omega : E\longrightarrow T^*\mathbb{R}^n$ is a 1-form if $\pi\circ\omega = id$.



$$\begin{tikzcd} T^*\mathbb{R}^n \arrow[d, shift left, "\pi"] \\ \mathbb{R}^n \arrow[u, shift left, "\omega"] \end{tikzcd}$$



<u>Example:</u> Let $X : E\longrightarrow T\mathbb{R}^n$ be a vector field.
Then $\omega : E\longrightarrow T^*\mathbb{R}^n$, $\omega_p(v) = \langle v, X(p)\rangle$,
is a 1-form.
<u>Pf:</u> For fixed p, $v\longmapsto \langle v, X(p)\rangle$ is linear map
$T_p\mathbb{R}^n \longrightarrow \mathbb{R}$ (we identify $\{p\}\times\mathbb{R}^n$ and $\mathbb{R}^n$)
Thus $\omega_p \in T_p^*\mathbb{R}^n$. $\square$

<u>Exercise:</u> If $\omega : E\longrightarrow T^*\mathbb{R}^n$ is a 1-form, then there exists a vector field $X : E\longrightarrow T\mathbb{R}^n$ so that $\omega_p(v) = \langle v, X(p)\rangle$.

<u>Observation:</u> Suppose $X : E\to T\mathbb{R}^n$ is a vector field
and $\omega : E\to T^*\mathbb{R}^n$ is a 1-form.
Then $p\longmapsto \omega_p(X(p))$ is a function $E\to\mathbb{R}$.

<u>Convention:</u> Given a 1-form $\omega : E\to T^*\mathbb{R}^n$, $p\in\mathbb{R}^n$, and $v\in T_p\mathbb{R}^n$
we use following notations
$$\omega_p(v) = \underbrace{\omega(p,v)}_{} = \underbrace{\omega(p)}_{}(v)$$

commonly used notation

emphasizes that ω depends on p and v

emphasizes that $\omega(p)$ is a linear map

Measurability

Let $(e_1 \dots e_n)$ be a standard basis in $\mathbb{R}^n$, and $\varepsilon_1 \dots \varepsilon_n$ the dual basis in $(\mathbb{R}^n)^*$.

We denote vectorfield $\mathbb{R}^n \to T\mathbb{R}^n$, $p \mapsto (p, e_k)$, simply by e_k and we denote by $\varepsilon_k : \mathbb{R}^n \to T^*\mathbb{R}^n$ the 1-form $\varepsilon(p)((p,v)) = \langle v, e_k \rangle$.

Lemma: Given a vectorfield $X : E \to T\mathbb{R}^n$ there exist functions $a_1 \dots a_n : E \to \mathbb{R}$ so that $X = \sum_{k=1}^{n} a_k e_k$.
Similarly given a 1-form $\omega : E \to T^*\mathbb{R}^n$ there exist functions $b_1 \dots b_n : E \to \mathbb{R}$ so that $\omega = \sum_{k=1}^{n} b_k \varepsilon_k$.

Pf: Exercise. □

Def: 1-form $\omega : E \to T^*\mathbb{R}^n$, $\omega = \sum b_k \varepsilon_k$, is
(a) measurable if E is measurable and functions b_k are measurable
(b) continuous if functions b_k are continuous
(c) C^1 if E is open and $b_k \in C^1$
(d) $C^2, C^3, \dots, C^\infty$ if b_k are $C^2, C^3, \dots, C^\infty$
(e) Lip if b_k are Lipschitz (and so on...)

Def: Suppose ω is continuous 1-form in $\Omega \subset \mathbb{R}^n$, $\gamma : [a,b] \to \Omega$ a C^1-path. We define
$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))\, dt.$$

Example: derivative of a function

Let $f : E \to \mathbb{R}$ be C^1 (and $E \subset \mathbb{R}^n$ open)

Then
$$\nabla f(p) = \left(\frac{\partial f}{\partial x_1}(p), \dots, \frac{\partial f}{\partial x_n}(p)\right) \underset{\text{redefine}}{=} \left(p, \frac{\partial f}{\partial x_1}(p) \dots \frac{\partial f}{\partial x_n}(p)\right) \in T_p\mathbb{R}^n.$$

Thus $\nabla f : E \to T\mathbb{R}^n$ is a vectorfield.

Def: $df : E \to T^*\mathbb{R}^n$ by
$$df_p(v) = \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t} = \underbrace{D_v f(p)}_{\text{directional derivative}}$$

Observation: $df_p(v) = \langle v, \nabla f(p) \rangle$ by prop. of gradient.

Observation: Let $\gamma : [a,b] \to \mathbb{R}^n$ C^1 path and $f : E \to \mathbb{R}$ C^1-function where $E \supset \gamma[a,b]$. Then

$$\int_\gamma df = \int_a^b df(\dot\gamma(t))\,dt = \int_a^b \langle \dot\gamma(t), \nabla f \rangle\,dt$$

$$= \int_a^b (f\circ\gamma)'(t)\,dt = f(b) - f(a).$$

Lemma: Let $x_i : \mathbb{R}^n \to \mathbb{R}$ be the projection $(x_1,\dots,x_n) \mapsto x_i$. Then $dx_i = \varepsilon_i$.

Pf: Exercise.

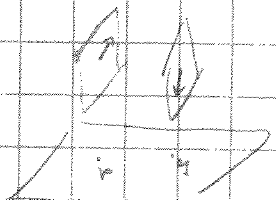
Cor: If ω is a 1-form then $\omega = \sum_{i=1}^n b_i\,dx_i$, where b_i is a function.

Why 1-forms:

Let $E \subset \mathbb{R}^n$ open and $f : E \to \mathbb{R}^m$ a C^1-map.
Redefine $Df_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$ $(p,v) \mapsto (f(p), (Df)_p v)$

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Suppose $X : E \to T\mathbb{R}^n$ is a vector field.

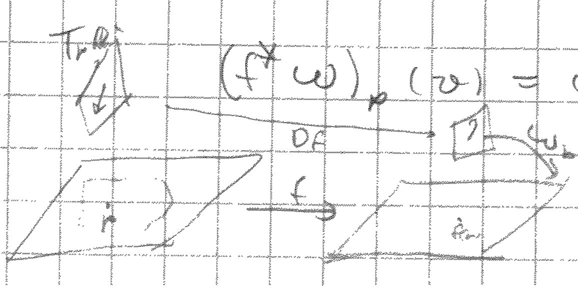


If f is injective we may define $f_*X : fE \to T\mathbb{R}^m$ by

$$(f_*X)_y = Df_{f^{-1}(y)}\, X(f^{-1}(y))$$

Def: Let $\omega : fE \to T^*\mathbb{R}^m$ 1-form. We define $f^*\omega : E \to T^*\mathbb{R}^n$

$$(f^*\omega)_p(v) = \omega_{f(p)}((Df)_p v)$$



$f^*\omega$ is called the pull-back of ω (by f)