Integration on manifolds

We begin with a Euclidean fact.

Lemma 1: (Change of variables) Let $f: U \to V$ be diffeomorphism, $U, V \subset \mathbb{R}^n$ open connected. Then

(1) $$\int_U f^* \omega = (\operatorname{sign} J_f) \int_V \omega$$ (See Rmk end of page)

for every $\omega \in \Omega^n_c(V)$.

Pf: Let $\omega = u \, dx_1 \wedge \cdots \wedge dx_n$. The Left Hand Side of (1) is

$$\int_U (u \circ f)(x) J_f(x) \, d\lambda^n = (\operatorname{sign} J_f) \int_U (u \circ f) |J_f| \, d\lambda^n$$

but by change of variables, $\int_U (u\circ f) |J_f| \, d\lambda^n = \int_V u \, d\lambda^n$; see [Rudin]. □

Def: (1) We say that an atlas $\mathcal{A} = \{(U_\alpha, x_\alpha) : \alpha \in I\}$ is *positive* if whenever $U_\alpha \cap U_\beta \neq \emptyset$ we have

$$J_{x_\beta \circ x_\alpha^{-1}}(q) > 0 \quad \text{for } q \in x_\alpha(U_\alpha \cap U_\beta).$$

(2) If M has a positive atlas $\mathcal{A}$, then we say that M is *orientable* and that $(M, \mathcal{A})$ is *oriented* manifold.

Rmk: Jacobian of a diffeo $f: U \to V$ does change sign if U connected: If J_f changes sign then $J_f(x) = 0$ for some $x \in U$. If $J_f(x) = 0$ then $Df(x)$ is not invertible but $Id = D(f^{-1} \circ f) = ((Df^{-1}) \circ f)(Df)$ contradiction □

Integration of compactly supported forms

Suppose $\omega \in \Omega^n_c(M)$ is such that $\mathrm{spt}\,\omega \subset U_\alpha$ where (U_α, x_α) is a chart.

We define $\displaystyle\int_M \omega \left(:= \int_{U_\alpha} \omega\right) = \int_{\mathbb{R}^n} (x_\alpha^{-1})^* \omega$

where integral in $\mathbb{R}^n$ is w.r.t. standard orientation $e_1, \ldots, e_n$

__Observation:__ By change of variables lemma, $\int_M \omega$ is independent of the choice of (U_α, x_α).

__Def:__ Let $(M, \mathcal{A})$ be an oriented n-manifold, $\mathcal{A} = \{(U_\alpha, x_\alpha) : \alpha \in I\}$, and $\omega \in \Omega^n_c(M)$. We define

$$\int_M \omega = \sum_k \int_M \varphi_k \omega$$

where $\{\varphi_k\}$ is a partition of unity w.r.t. $\{U_\alpha\}_{\alpha \in I}$.

__Lemma:__ $\int_M \omega$ is independent on the partition of unity.

__Pf:__ Suppose $\{\psi_j\}$ is another PU. Then $\sum_j \psi_j \omega$ and $\sum_k \varphi_k \omega$ are finite sums and

$$\sum_k \int_M \varphi_k \omega = \sum_k \int_M \varphi_k \Big(\sum_j \psi_j \omega\Big) = \sum_k \sum_j \int_M \varphi_k \psi_j \omega$$

$$= \sum_j \int_M \psi_j \Big(\sum_k \varphi_k \omega\Big) = \sum_j \int_M \psi_j \omega.$$

$\square$

Rmk 1: The integral $\int_M \omega$ depends on the orientation of M!

Suppose $\mathcal{A}$ and $\mathcal{A}'$ are positive atlases on M so that $\mathcal{A} \cup \mathcal{A}'$ is not positive.
Then there exists a cover $\{U_i\}$ of M and charts $(U_i, x_i) \in \mathcal{A}$, $(U_i, y_i) \in \mathcal{A}'$ so that

$$J_{y_i \circ x_i^{-1}} < 0 \quad \text{for every } i.$$

Let $\{\varphi_i\}$ be a partition of unity w.r.t $\{U_i\}$ and $\omega \in \Omega^n_c(M)$. Then

$$\int_{(M,\mathcal{A}')} \omega = \sum_i \int_{(M,\mathcal{A}')} \varphi_i \omega = \sum_i \int_{y_i(U_i)} (y_i^{-1})^*(\varphi_i \omega)$$

$$= \sum_i \operatorname{sign}(J_{y_i \circ x_i^{-1}}) \int_{x_i(U_i)} (y_i \circ x_i^{-1})^* (y_i^{-1})^* (\varphi_i \omega)$$

$$= - \sum_i \int_{x_i(U_i)} (x_i^{-1})^* (\varphi_i \omega) = - \int_{(M,\mathcal{A})} \omega .$$

Def: A smooth mapping $f: M \to N$ between oriented n-manifolds $(M, \mathcal{A})$ and $(N, \mathcal{A}')$ is orientation preserving if

$$J_{y \circ f \circ x^{-1}} > 0 \quad \text{on } xU$$

whenever

$$(U, x) \in \mathcal{A}, \ (V, y) \in \mathcal{A}' \ \text{and} \ fU \subset V.$$

If

$$J_{y \circ f \circ x^{-1}} < 0 \quad \text{on } xU$$

then we say that f is orientation reversing

Rmk 2: Rmk 1 $\Rightarrow$ $\int_N f^* \omega = \int_M \omega$ if $f: M \to N$ orientation preserving diffeo

and $\int_N f^* \omega = - \int_M \omega$ if f orientation reversing diffeo.

Def: Oriented manifolds $(M, \mathcal{A})$ and $(M, \mathcal{A}')$ are equivalent if $id: M \to M$ is orientation preserving

Two main theorems on integration:

Thm: (Stokes)

Suppose M is an oriented n-manifold with boundary ∂M. Then

$$\int_M d\omega = \int_{\partial M} \omega \qquad \text{for every } \omega \in \Omega_c^{n-1}(M).$$

Thm: (Fantastic II) For an oriented and connected smooth n-manifold M, the sequence

$$\Omega_c^{n-1}(M) \xrightarrow{d} \Omega_c^n(M) \xrightarrow{\int_M} \mathbb{R} \to 0$$

is exact. In particular, for $\omega \in \Omega_c^n(M)$,

$$\int_M \omega = 0 \iff \omega \text{ is exact.}$$

Cor: If M is oriented and connected

$$\int_M : H_c^n(M) \longrightarrow \mathbb{R}$$

is an isomorphism. In particular if M is also compact, then

$$\int_M : H^n(M) \to \mathbb{R}$$

is an isomorphism.

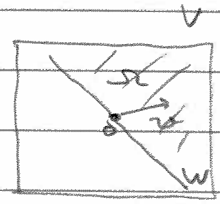
Comparison: Orientation on vector spaces and on manifolds

Recall: Let V be n-dimensional vector space
An n-vector $\xi \in \Lambda_n V$ $(\cong \mathbb{R})$ is an orientation of V if $\xi \neq 0$.

Let $W \subset V$ be a codim 1 subspace i.e. $\dim W = \dim V - 1 = n-1$ and ζ an orientation of W. Let Ω be a component of $V \setminus W$. Then (W, ζ) is positively oriented w.r.t. Ω i (V, ξ) if $\exists\, v \in \Omega$ s.t.

$$v \wedge \zeta = \lambda \xi$$

where $\lambda > 0$.



Dual definitions: $\omega \in Alt^n(V)$ is an orientation of V if $\omega \neq 0$

Suppose $\tau \in Alt^{n-1}(W)$ is an orientation on W. Then (W, τ) is positively oriented w.r.t. Ω i (V, ω) if $\exists\, v \in \Omega$ and $\lambda > 0$ s.t.

$$\tau(w_1, \ldots, w_{n-1}) = \lambda\, \omega(v, w_1, \ldots, w_{n-1})$$

for all $w_1, \ldots, w_{n-1}$.

Rmk: For $\sigma \in Alt^k(V)$, $v \in V$, the map $v \lrcorner \sigma \in Alt^{k-1}(V)$,

$$v \lrcorner \sigma(v_1, \ldots, v_{k-1}) = \sigma(v, v_1, \ldots, v_{k-1}),$$

is called the contraction of σ by v.

Rmk: If $\xi \in \Lambda^n V$ and $\omega \in Alt^n(V)$ are orientations then $\omega(\xi) \neq 0$, where $\omega : \Lambda^n V \to \mathbb{R}$ is the linearization of ω.

Def. We say that (V, ξ) and (V, ω) are similarly oriented if $\omega(\xi) > 0$.

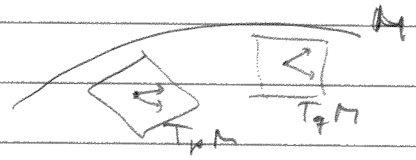
Orientation of a manifold and orientation in vector spaces is related by following Thm...

<u>Thm:</u> An n-manifold M has a positive atlas if and only if there exists non-vanishing $\omega \in \Omega^n(M)$, i.e. $\omega_p \neq 0$ for every $p \in M$.

<u>Pf:</u> Exercise 12/4

<u>Comment:</u> So a manifold is orientable iff there exists an orientation $\omega_p \in Alt^n(T_pM)$ that varies smoothly on M

<u>Def.</u> Let M be a smooth n-manifold. We call non-vanishing forms ω as <u>orientation forms</u>. Form ω is <u>compatible</u> with positive atlas $\mathcal{A} = \{(U_i, x_i) : i \in I\}$ if $(x^{-1})^*\omega = \lambda \, dx_1 \wedge \dots \wedge dx_n$ for $\lambda \in C^\infty(xU)$ positive.

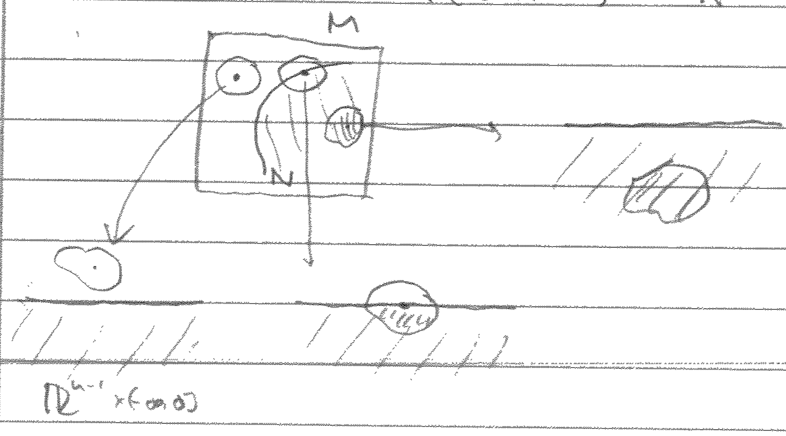


<u>Domains with smooth boundary</u>

Let M be a smooth n-manifold

<u>Def:</u> A subset $N \subset M$ is a <u>domain with smooth boundary</u> if for every $p \in \underline{M}$ there exists a chart (U, x) of M at p so that

(1) $$x(U \cap N) = xU \cap (\mathbb{R}^{n-1} \times (-\infty, 0])$$



<u>Rmk:</u> (1) If $N \neq \emptyset$ then $\operatorname{int} N \neq \emptyset$

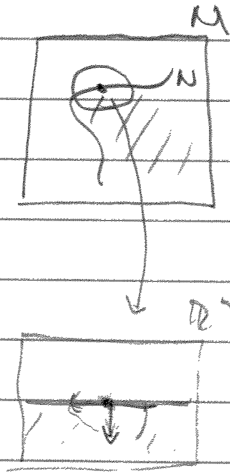
(2) ∂N is a smooth $(n-1)$-manifold.

Let $N \subset M$ be a domain with smooth boundary.
Let $\omega \in \Omega^n(M)$ be an orientation form on M and
let $\tau \in \Omega^{n-1}(\partial N)$ be an orientation form on N.

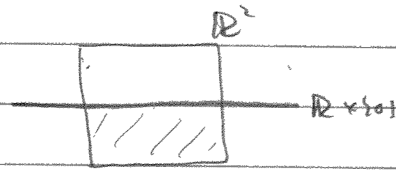
Def: $(\partial N, \tau)$ is positively $\overset{(\text{neg.})}{\text{oriented}}$ w.r.t. N in (M, ω) if for every $p \in \partial N$ and every chart (U, x) satisfying (1) we have

$$(2) \qquad \left((x|U\cap\partial N)^{-1}\right)^* \tau = \lambda \left((\pm e_n) \lrcorner (x^{-1})^* \omega\right)$$

where $\lambda \in C^\infty(x(U \cap \partial N))$ is positive (neg)



Example:



$(\mathbb{R}^2, dx_1 \wedge dx_2) \qquad (\mathbb{R}\times\{0\}, dx_1)$

$$(-e_2) \lrcorner (dx_1 \wedge dx_2)(v) = dx_1 \wedge dx_2(-e_2, v)$$
$$= -dx_2(-e_2)\, dx_1(v) = dx_1(v)$$

Thus $(\mathbb{R}\times\{0\}, dx_1)$ is pos. oriented w.r.t. $\mathbb{R}^2_-$ in $(\mathbb{R}^2, dx_1 \wedge dx_2)$.

Excursion: Let $p \in \partial N$ and $v \in T_pM$.
We say that v is inward directed if
$$\langle Dx(v), e_n \rangle < 0$$
and outward directed if
$$\langle Dx(v), e_n \rangle > 0$$
when (U, x) is a chart satisfying (1).
Rmk: condition independent of the choice of (U, x).

Rmk: We use inward directed tangent vectors in the definition of positively oriented boundary.
~~The use of outward directed is more common~~

Orientation preserving mappings and forms

Suppose M and N are orientable n-manifolds with positive atlases $\mathcal{A}$ and $\mathcal{A}'$, respectively, and compatible orientation forms ω_M and ω_N, respectively.

<u>Lemma:</u> A smooth map $f: M \to N$ is orientation preserving if and only if $f^*\omega_N = \lambda\,\omega_M$ where $\lambda \in C^\infty$ is positive.

<u>Pf:</u> As exercise 12/4 $\square$

Stokes' thm: Let M be a smooth oriented n-manifold. Let $N \subset M$ be a domain with smooth boundary. Suppose that ∂N is negatively oriented w.r.t. N int (i.e. pos oriented w.r.t. $M \setminus N$). Then

$$\int_N d\omega = \int_{\partial N} i^*\omega \quad \text{for } \omega \in \Omega_c^{n-1}(M).$$

where $i : \partial N \to N$ is the inclusion.

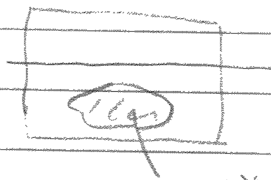
Rmk: If $\partial N = \emptyset$ then $\int_{\partial N} \omega = 0$ (see part $(*)$ in the proof)

Thus $\int_M d\omega = 0 \quad \forall\, \omega \in \Omega_c^{n-1}(M)$.

Proof: Let $\mathcal{A} = \{(U_i, x_i) : i \in I\}$ be a positive atlas on M so that $x_i(U_i \cap N) = x_i U_i \cap \mathbb{R}^{n-1} \times (-\infty, 0] \quad \forall i$.
Let $\{\varphi_i\}$ be a partition of unity w.r.t. $\mathcal{A}$.

Let $\omega \in \Omega_c^{n-1}(M)$.

If $\operatorname{spt} (x_i^{-1})^*(\varphi_i \omega) \subset \mathbb{R}^{n-1} \times (-\infty, 0)$, then $(x_i^{-1})^*(\varphi_i\omega) \in \Omega_c^{n-1}(\mathbb{R}^n)$.
Thus, by the Fantastic thm,



$\operatorname{spt} (x_i^{-1})^*(\varphi_i\omega)$

$$\int_{\mathbb{R}^n} d\left((x_i^{-1})^* \varphi_i \omega\right) = 0$$

If $\operatorname{spt} (x_i^{-1})^*(\varphi_i\omega) \cap \mathbb{R}^{n-1} \times \{0\} \neq \emptyset$, then, by the baby Stokes' thm,

$$\int_{(\mathbb{R}^{n-1}\times(-\infty,0))\cap x_i U_i} d\left((x_i^{-1})^* \varphi_i \omega\right) = -\lambda \int_{(\mathbb{R}^{n-1}\times\{0\})\cap x_i U_i} (x_i^{-1})^* \varphi_i \omega$$

where $\mathbb{R}^n$ is oriented with $e_1 \wedge \dots \wedge e_n$, $\mathbb{R}^{n-1}\times\{0\}$ with $e_1 \wedge \dots \wedge e_{n-1}$ and the constant $\lambda \in \{\pm 1\}$ is given by

$$(-e_n) \wedge (e_1 \wedge \cdots \wedge e_{n-1}) = \lambda \; e_1 \wedge \cdots \wedge e_n.$$

($\lambda = 1$ iff $\mathbb{R}^{n-1} \times \{0\}$ is positively oriented w.r.t. $\mathbb{R}^{n-1} \times (-\infty, 0]$ in $\mathbb{R}^n$)

Thus $\lambda = -(-1)^{n-1} = (-1)^n$.

Let $\omega_M \in \Omega^n(M)$ and $\tau \in \Omega^{n-1}(\partial N)$ be orientation forms compatible with chosen orientations on M and ∂N, respectively. Then $(x_i^{-1})^* \omega_M = u \, dx_1 \wedge \cdots \wedge dx_n$ where $u \in C^\infty$ pos.
Since ∂N is neg. oriented, $\exists \, \nu \in C^\infty$ negative s.t. that

$$\left((x_i | U_i \cap \partial N)^{-1} \right)^* \tau = \nu \, (-e_n) \lrcorner \, (x_i^{-1})^* \omega_M$$
$$= -\nu u \; e_n \lrcorner \, dx_1 \wedge \cdots \wedge dx_n$$
$$= -\nu u \, (-1)^{n-1} \, dx_1 \wedge \cdots \wedge dx_{n-1}$$
$$= \nu u \, (-1)^n \, dx_1 \wedge \cdots \wedge dx_{n-1}$$
$$= -(-1)^n \, (-\nu u) \, dx_1 \wedge \cdots \wedge dx_{n-1}$$

Thus $(x_i | U_i \cap \partial N)^{-1}$ is orientation preserving if $-(-1)^n = 1$.
Hence

$$\int_{U_i \cap \partial N} i^*(\psi_i \omega) = -(-1)^n \int_{x_i(U_i \cap \partial N)} \left((x_i | U_i \cap \partial N)^{-1} \right)^* \psi_i \omega$$

$$= -(-1)^n \int_{(\mathbb{R}^{n-1} \times \{0\}) \cap x_i U_i} \left((x_i | U_i \cap \partial N)^{-1} \right)^* \psi_i \omega$$

$$= \int_{(\mathbb{R}^{n-1} \times (-\infty, 0]) \cap x_i U_i} d \left((x_i | U_i \cap \partial N)^{-1} \right)^* \psi_i \omega$$

$$= \int_{U_i \cap N} d(\psi_i \omega).$$

Thus $$\int_{\partial N} i^* \omega = \sum_i \int_{\partial N} i^*(\psi_i \omega) = \sum_i \int_N d(\psi_i \omega) = \int_N d\omega$$

$$\sum_i d(\psi_i \omega) = d\left(\sum_i \psi_i \omega \right) = d\omega.$$

□

Thm: (Fantastic II)

For an oriented and connected smooth n-manifold M, the sequence

$$\Omega_c^{n-1}(M) \xrightarrow{d} \Omega_c^n(M) \xrightarrow{\int_M} \mathbb{R} \to 0$$

is exact. In particular, for $\omega \in \Omega_c^n(M)$,

$$\int_M \omega = 0 \iff \omega \text{ is exact.}$$

Rmk: Since $\int_M d\omega = 0$ for all $\omega \in \Omega_c^{n-1}$ by Stokes we have
$$\underbrace{\Omega_c^{n-1}(M) \xrightarrow{d} \Omega_c^n(M) \xrightarrow{\int_M}}_{=0}$$

For the proof we need

Thm: (Fantastic) For $\omega \in \Omega_c^n(\mathbb{R}^n)$, we have $\int_{\mathbb{R}^n} \omega = 0$ iff ω is exact

i.e. $$\Omega_c^{n-1}(\mathbb{R}^n) \xrightarrow{d} \Omega_c^n(\mathbb{R}^n) \xrightarrow{\int_{\mathbb{R}^n}} \mathbb{R} \to 0$$
is exact.

and following lemmas.

Lemma 1: Let $(U_\alpha)_{\alpha \in I}$ be an open cover of a connected manifold M and let $p, q \in M$. Then exists $\alpha_1 \dots \alpha_k \in I$ s.t.

(i) $p \in U_{\alpha_1}$, $q \in U_{\alpha_k}$

(ii) $U_{\alpha_i} \cap U_{\alpha_{i+1}} \neq \emptyset$ for $1 \leq i \leq k-1$.

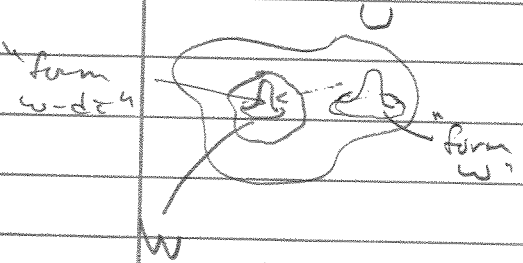
Pf: Let $p \in M$.

Define $M_p = \{ q \in M : \exists$ seq. $\alpha_1 \dots \alpha_k$ satisfying (i) and (ii)$\}$

Then M_p is open (if $q \in M_p$ and $\alpha_1 \dots \alpha_k$ is the seq, then $U_{\alpha_k} \subset M_p$) and closed (if $\tilde{q} \in \partial M_p$ then $\exists U_\alpha$ s.t. $\tilde{q} \in U_\alpha$ and $q \in M_p \cap U_\alpha$. Then $U_{\alpha_1} \dots U_{\alpha_k}, U_\alpha$ is a valid seq. Thus $\tilde{q} \in M_p$.)

Since M is connected, $M_p = M$. $\square$

Lemma 2: (Lemma of sliding hump) Suppose $U \subset M$ is diffeo to $\mathbb{R}^n$ and let $W \subset U$ be non-empty and open. Then for every $\omega \in \Omega^n_c(U)$ there exists $\tau \in \Omega^{n-1}_c(U)$ such that $\omega - d\tau \in \Omega^n_c(W)$



Pf: Let $\varphi : U \to \mathbb{R}^n$ be a diffeo.
Let $\tilde{\omega} = (\varphi^{-1})^* \omega$. and $\tilde{W} = \varphi W$.

Fix $u \in C^\infty_0(\tilde{W})$ s.t. $\int_{\tilde{W}} u \, d\alpha^n = 1$

Since φ is homeo, $\tilde{\omega}$ is compactly supported.
Thus $\lambda = \int_{\mathbb{R}^n} \tilde{\omega}$ is well-defined and finite.

Then $\int_{\mathbb{R}^n} (\lambda u \, dx_1 \wedge \dots \wedge dx_n - \tilde{\omega}) = \lambda \int_{\mathbb{R}^n} u \, d\alpha^n - \int_{\mathbb{R}^n} \tilde{\omega} = 0.$

By Fantastic Thm, there exists $\tilde{\tau} \in \Omega^{n-1}_c(\mathbb{R}^n)$
s.t. $d\tilde{\tau} = \tilde{\omega} - \lambda u \, dx_1 \wedge \dots \wedge dx_n$

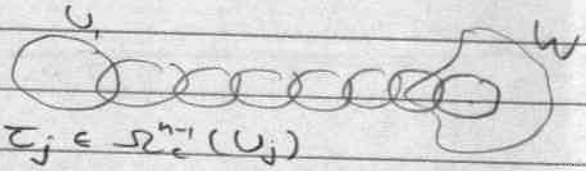
Let $\tau = \varphi^* \tilde{\tau} \in \Omega^{n-1}_c(U)$.
Then

$$\begin{aligned}
\omega - d\tau &= \omega - d\varphi^* \tilde{\tau} \\
&= \omega - \varphi^* d\tilde{\tau} \\
&= \omega - \varphi^*(\tilde{\omega} - \lambda u \, dx_1 \wedge \dots \wedge dx_n) \\
&= \omega - \omega + \lambda \varphi^*(u \, dx_1 \wedge \dots \wedge dx_n) \\
&= \lambda \varphi^*(u \, dx_1 \wedge \dots \wedge dx_n) \in \Omega^n_c(W)
\end{aligned}$$

$\square$

Lemma 3: (Lemma of chopping up) Suppose M is connected and $W \subset M$ non-empty and open. Then for every $\omega \in \Omega^n_c(M)$ there exists $\tau \in \Omega^{n-1}_c(M)$ s.t. $\omega - d\tau \in \Omega^n_c(W)$.

Pf: Suppose first that $\omega \in \Omega^n_c(U_1)$ where U_1 diffeo to $\mathbb{R}^n$.
By Lemma 1, there exists a seq. $U_1 \dots U_k$ of open sets s.t. U_j diffeo to $\mathbb{R}^n$, $U_j \cap U_{j+1} \neq \emptyset$ $\forall_{1 \le j \le k}$ and $U_k \subset W$.
Apply Lemma 2 iteratively to find forms $\tau_j \in \Omega^{n-1}_c(U_j)$ so that $\omega - \sum_{i=1}^{j} d\tau_i \in \Omega^n_c(U_j)$



Then $\tau = \sum_{i=1}^{k} \tau_i$ satisfy the claim.

Suppose now that $\omega \in \Omega^n_c(M)$. By compactness there exists a finite cover $\{V_1 \dots V_m\}$ of spt ω by open sets diffeomorphic to $\mathbb{R}^n$. Let $\{\psi_i\}$ be a partition of unity w.r.t. $\{V_1 \dots V_m\}$ on $\bigcup_{i=1}^{m} V_i$.
Since $\omega = \sum_i \psi_i \omega$,
there exists forms $\tau_i \in \Omega^{n-1}_c(M)$ s.t.
$\psi_i \omega - d\tau_i \in \Omega^n_c(W)$.
Let $\tau = \sum_i \tau_i$. Then $\tau \in \Omega^{n-1}_c(M)$ and

$$\omega - d\tau = \sum_i \psi_i \omega - d\sum_i \tau_i = \sum_i \psi_i \omega - d\tau_i \in \Omega^n_c(W)$$

$\square$

Pf of Fantastic Thm: $\int_M : \Omega^n_c(M) \longrightarrow \mathbb{R}$ surjective (ok)
Suppose $\omega \in \Omega^n_c(M)$ satisfies $\int_M \omega = 0$. Take $W \subset M$ open and diffeo to $\mathbb{R}^n$.
Fix $\tau \in \Omega^{n-1}_c(M)$ s.t. $\omega - d\tau \in \Omega^n_c(W)$. Then
$$\int_W (\omega - d\tau) = \int_M \omega - \int_M d\tau = 0 \quad \text{(by Stokes)}$$
Thus $\omega - d\tau$ is exact (by Fantastic Thm). Hence
$\omega = d\tau - d\kappa$ where $\kappa \in \Omega^{n-1}_c(W)$ $\square$

Degree of a proper map.

Def: A smooth map $f: M \to N$ is *proper* if $f^{-1}K$ is compact for every $K \subset N$ compact.

Lemma: If $f: M \to N$ proper then $f^*: H^k_c(N) \to H^k_c(M)$ well-defined.

Pf: Let $\omega \in \Omega^k_c(N)$ be closed. Then $\operatorname{spt} \omega$ is compact. Then $\operatorname{spt} f^*\omega$ is compact and $f^*\omega$ is closed. Thus $[f^*\omega] \in H^k_c(M)$ for $[\omega] \in H^k_c(N)$. $\square$

Suppose M and N are orientable connected n-manifolds. Then $H^n_c(M) \cong H^n_c(N) \cong \mathbb{R}$. Thus there exists a linear map

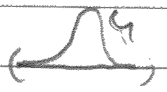
$$\begin{tikzcd}
H^n_c(N) \arrow[r, "f^*"] \arrow[d, "\int_N"'] \arrow[dr, phantom, "\circlearrowleft"] & H^n_c(M) \arrow[d, "\int_M"] \\
\mathbb{R} \arrow[r] & \mathbb{R} \\[-20pt]
x \arrow[r, mapsto] & \lambda x
\end{tikzcd}$$

Def: λ is the degree $\deg f$ of f.

Comment: $\deg f$ depends on cohomology rings $H^*(M)$ and $H^*(N)$.

Thm: $\deg(f) = 0$ for every continuous map $f: S^2 \to S^1 \times S^1$.

Pf: By homotopy invariance, may assume f smooth.

We fix $u \in C^\infty_c((-1,1))$ s.t. $\int u = 1$ 
Let (U, φ) be a chart on S^1 s.t. $\varphi U = (-1,1)$.
Let $\tau = \varphi^*(u\,dx)$. Then $[\tau]$ spans $H^1(S^1)$.
Let $\pi_i: S^1 \times S^1 \to S^1$, $(p_1, p_2) \mapsto p_i$ and $\omega_i = \pi_i^* \tau$.

Then $\omega_1 \wedge \omega_2 \in \Omega^2(S^1 \times S^1)$ and $\operatorname{spt} \omega_1 \wedge \omega_2 \subset U \times U$.
Since $(U \times U, \varphi \times \varphi)$ is a chart on $S^1 \times S^1$ we have

$$\int_{S^1 \times S^1} \omega_1 \wedge \omega_2 = \int_{(-1,1)^2} u(x_1)\, u(x_2)\, dx_1 \wedge dx_2 = 1.$$

(assuming $(U \times U, \varphi \times \varphi)$ pos. chart.)

Thus $[\omega_1 \wedge \omega_2] \neq 0$ in $H^2(S^1 \times S^1)$

However, $[f^* \omega_1] = [f^* \omega_2] = 0$ since $H^1(S^2) = 0$.
Thus $\exists\, \eta \in \Omega^0(S^2)$ s.t. $f^* \omega_1 = d\eta$.

Hence $f^*(\omega_1 \wedge \omega_2) = f^* \omega_1 \wedge f^* \omega_2 = d\eta \wedge f^* \omega_2$.
Since $f^* \omega_2$ is closed, $d(\eta \wedge f^* \omega_2) = d\eta \wedge f^* \omega_2 + \eta\, d f^* \omega_2$
$= d\eta \wedge f^* \omega_2$.

Thus

$$\int_{S^2} f^*(\omega_1 \wedge \omega_2) = \int_{S^2} f^* \omega_1 \wedge f^* \omega_2 = \int_{S^2} d\eta \wedge f^* \omega_2$$

$$= \int_{S^2} d(\eta \wedge f^* \omega_2) = 0 \quad \text{by Stokes.}$$

Since $\int_{S^1 \times S^1} \omega_1 \wedge \omega_2 = 1$ we have $\deg f = 0$.

$\square$

Rmk: This is a special case of a more general argument using Poincaré duality. $\square$

The End.