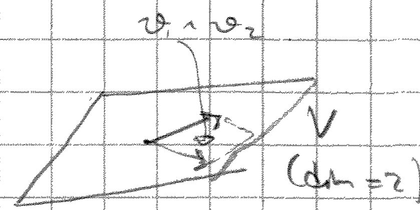
Orientation



Let V be k-dim vector space.

Then $\mathrm{Alt}^k(V)$ and $\Lambda_k V$ are 1-dimensional.

Definition: A non-zero $\xi \in \Lambda_k V$ and $\omega \in \mathrm{Alt}^k(V)$ are called orientation k-vector and orientation k-covector of V, respectively. Both are also called volume elements of V. The pairs (V, ξ) and (V, ω) are called oriented vector spaces.

Suppose $W \subset V$ subspace $\dim W = l < k$.
Let $i : W \hookrightarrow V$ be the inclusion. Then
$i_* : \Lambda_l W \to \Lambda_l V$ and $i^* : \mathrm{Alt}^l(V) \to \mathrm{Alt}^l(W)$
are injective and surjective. (From now on: $\Lambda_l W \subset \Lambda_l V$)

Def: Let ξ be an orientation k-vector of W.
The vector $i_* \xi$ is a direction of W in V.

Observation: $\Lambda_l W$ is 1-dim $\Rightarrow$ $i_* \Lambda_l W$ is 1-dim
(compare to the case of a line in $\mathbb{R}^n$).

Affine subspaces (see expanded version after orientation)

Def: $P \subset V$ is an affine subspace if $\exists\, p \in P$ and (unique) $W \subset V$ subspace so that $P = p + W = \{p + w : w \in W\}$.

We set (formally) $\Lambda_k P := \Lambda_k W$
$\mathrm{Alt}^k(P) = \mathrm{Alt}^k(W)$
$\dim P = \dim W$

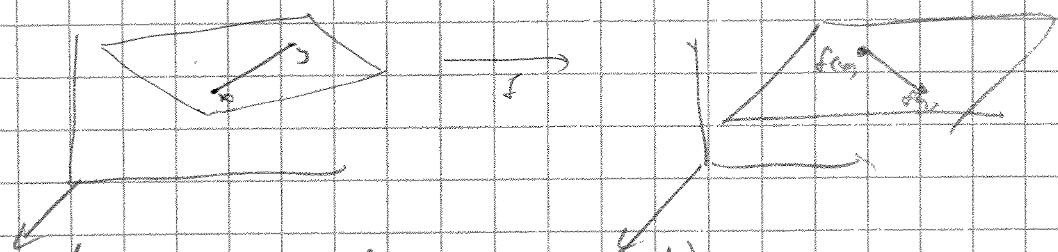
Also: (P, ξ) is an oriented affine subspace if $\xi \in \Lambda_{\dim W} W$ is non-zero.

A map $f : P \to Q$ between affine subspaces is affine if $P = p + W$ and $Q = q + W'$ and the map $W \to W'$, $w \mapsto f(p + w) - f(p)$ is linear.

Rmk: Intrinsic definition for affine maps:

$f: P \to Q$ affine if $f(tx + (1-t)y) = t f(x) + (1-t) f(y)$

for all $x, y \in P$, $t \in \mathbb{R}$



(Check that def. are equivalent!)

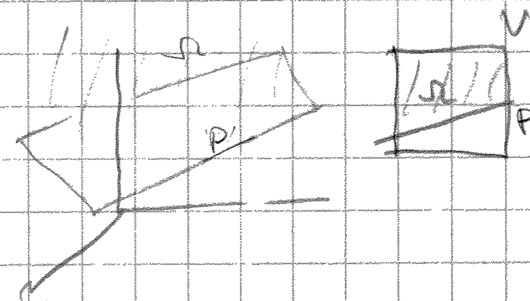
Let $P \subset V$ affine subspace, $\dim P = n-1$, $\dim V = n$.

Let Ω be a connected component of $V \setminus P$

(i.e. $\partial \Omega = W$)

Let ξ_P be an orientation of P

and ξ_V an orientation of V.



$= p + W$ $\quad \in \Lambda_{n-1} W$

Def: (P, ξ_P) is positively oriented in (V, ξ_V)

with respect to Ω if ($\Rightarrow v \neq 0$)

for every $p \in P$ and $v \in V$ so that $p + v \in \Omega$

we have $v \wedge \xi_P = \lambda \xi_V$

for $\lambda > 0$. (More precisely, $v \wedge c_v \xi_P = \lambda \xi_V$)

Otherwise, (W, ξ_W) is negatively oriented in (V, ξ_V) w.r.t. Ω

Lemma: The sign of λ does not depend on p and v.

i.e. if $p_0 \in P$, $v_0 \in V$ so that $p_0 + v_0 \in \Omega$ give

$v_0 \wedge \xi_P = \lambda_0 \xi_V$, where $\lambda_0 > 0$, then the same holds

for all pairs (p, v)

Pf: Let $(e_1, \dots, e_{n-1})$ be a basis of W and extend it to a

basis $(e_1, \dots, e_n)$ of V.

Let $p \in P$ and $p_0 \in P$ and $v, v_0 \in V$ so that $p + v, p_0 + v_0 \in \Omega$.

Let $v_0 = w_0 + c_0 e_n$ where $w_0 \in W$.

and $v = w + c\, e_n$ where $w \in W$.

Since $p + v \in \Omega$ and $p_0 + v_0 \in \Omega$, we have that

c and c_0 have the same sign, by continuity.

Since $\xi_p (= i_v \xi_p) = \alpha_p e_1 \wedge \cdots \wedge e_{n-1}$, $\alpha_p \neq 0$,
and $\xi_V = \alpha_V e_1 \wedge \cdots \wedge e_n$, we have

$$v_0 \wedge \xi_p = \alpha_p \underbrace{v_0 \wedge e_1 \wedge \cdots \wedge e_{n-1}}_{=0 \text{ by lin. dep.}} + \alpha_p c_0 e_n \wedge e_1 \wedge \cdots \wedge e_{n-1}$$

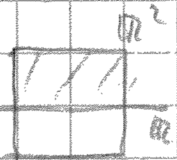
$$= \alpha_p c_0 (-1)^{n-1} e_1 \wedge \cdots \wedge e_n$$

$$= \frac{\alpha_p}{\alpha_V} c_0 (-1)^{n-1} \xi_V .$$

Similarly, $v \wedge \xi_p = \frac{\alpha_p}{\alpha_V} c (-1)^{n-1} \xi_V$

Thus $v \wedge \xi_p > 0$ if and only if $v_0 \wedge \xi_p > 0$. □

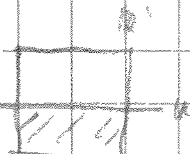
Examples: (1) Let $V = \mathbb{R}^2$ $\quad \xi_V = e_1 \wedge e_2$
$W = \mathbb{R} \times \{0\}$ $\quad \xi_W = e_1$



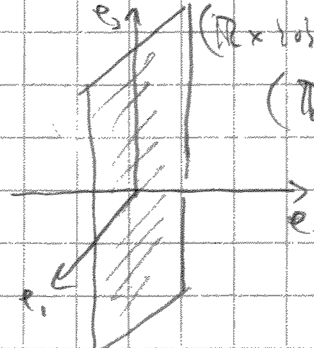
If $\Omega = \mathbb{R} \times (0, \infty)$ then take $p_0 = 0$, $v_0 = e_2$.
we get

$$v_0 \wedge \xi_W = e_2 \wedge e_1 = - e_1 \wedge e_2$$

Thus $(\mathbb{R} \times \{0\}, e_1)$ is neg oriented in $(\mathbb{R}^2, e_1 \wedge e_2)$
with respect to $\mathbb{R} \times (0, \infty)$.



If $\Omega = \mathbb{R} \times (-\infty, 0)$, then $(\mathbb{R} \times \{0\}, e_1)$ is pos oriented.



(2) $(\mathbb{R} \times \{0\} \times \mathbb{R}, e_1 \wedge e_3)$ neg. oriented in
$(\mathbb{R}^3, e_1 \wedge e_2 \wedge e_3)$ w.r.t. $\mathbb{R} \times (0, \infty) \times \mathbb{R}$.

□

Let $P \subset V$ be an affine subspace with orientation ξ_P
Let $E \subset P$ a subset
<u>Def:</u> We call (E, ξ_P) an oriented set

Let $f: P \longrightarrow Q$ be an affine map between oriented affine subspaces (P, ξ_P), (Q, ξ_Q) of vector spaces V and W, and $\dim P = \dim Q = k$
<u>Def:</u> The Jacobian of f is the number satisfying $f_* \xi_P = J_f \xi_Q$.
<u>Def:</u> f is <u>orientation preserving</u> if $J_f > 0$
f is <u>orientation reversing</u> if $J_f < 0$.

<u>Remark:</u> (How f_* is defined)
Let $f: P \to Q$ affine. Let $P = p + P_0$ and $Q = q + Q_0$
The map $g_f: P_0 \to Q_0$, $v \mapsto f(p+v) - f(p)$ is linear (and unique).
$\xi \in \Lambda_k P_0$ when $k = \dim P_0$
Then $f_* \xi_P$ is defined by $f_* \xi_P = (g_f)_* \xi_P$

Since $\Lambda_k Q_0$ is 1-dim and $\xi_Q \in \Lambda_k Q_0$ spans $\Lambda_k Q_0$,

$(g_f)_* \xi_P = J_f \xi_Q$.

Moreover, if $(e_1, \dots, e_k)$ is a basis of P_0 and $(e_1', \dots, e_k')$ is a basis of Q_0 then
$\lambda_f = \det A_f$ where A_f is the matrix of f in these bases.

<u>Rmk:</u> A matrix has a determinant, but a linear map doesn't unless orientations are fixed.

<u>Def:</u> Orientations ξ and J on P are <u>equivalent</u> if
$id: (P, \xi) \longrightarrow (P, J)$ is orientation preserving.
Equivalence class $[\xi]$ is also called as <u>orientation (class)</u> of P and $(P, [\xi])$ is also called an <u>oriented vectorspace</u>
$\mathcal{O}(P) = \{[\xi] : \xi \in \Lambda_k P, \xi \neq 0\}$ where $\dim P = k$.

Vector fields and forms II

k^{th} exterior bundle

Def: $\Lambda_k T\mathbb{R}^n = \bigcup_{p\in\mathbb{R}^n} \Lambda_k T_p\mathbb{R}^n$ is the k-th exterior (tangent) bundle of $\mathbb{R}^n$

Obs: $\Lambda_k T\mathbb{R}^n \cong \mathbb{R}^n \times \Lambda_k \mathbb{R}^n$; we have $\pi : \Lambda_k T\mathbb{R}^n \longrightarrow \mathbb{R}^n$
$\Lambda_k T_p\mathbb{R}^n \ni \xi \longmapsto p$.

Def: A map $X : E \to \Lambda_k T\mathbb{R}^n$ is a <u>k-vector field</u> if
$\pi \circ X = id$.

$$\begin{array}{c} \Lambda_k T\mathbb{R}^n \\ X \uparrow \downarrow \pi \\ E \subset \mathbb{R}^n \end{array}$$

Def: $Alt^k(T\mathbb{R}^n) = \bigcup_{p\in\mathbb{R}^n} Alt^k(T_p\mathbb{R}^n)$ is the k-th exterior cotangent bundle
we have $\pi : Alt^k(T\mathbb{R}^n) \longrightarrow \mathbb{R}^n$
$Alt^k(T_p\mathbb{R}^n) \ni f \longmapsto p$

Def: A map $\omega : E \to Alt^k(T\mathbb{R}^n)$ is a <u>k-form</u> if
$\pi \circ X = id$.

Lemma: If ω is a k-form then

$$\omega = \sum_{1\le i_1 < \dots < i_k \le n} b_{i_1 \dots i_k}\, dx_{i_1} \wedge \dots \wedge dx_{i_k} = \sum_I b_I\, dx_I$$

where $I = (i_1, \dots, i_k)$, $i_1 < \dots < i_k$ and $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}$.

Pf: $(dx_i)(p) = \varepsilon_i$ and $\varepsilon_{i_1} \wedge \dots \wedge \varepsilon_{i_k}$ form a basis for every $p \in \mathbb{R}^n$ $\square$

Notation: $\Gamma_k(E) = \Gamma(E, \Lambda_k T\mathbb{R}^n)$ = space of k-vector fields on E
$\Gamma^k(E) = \Gamma(E, Alt^k(T\mathbb{R}^n))$ space of k-forms on E

$U \subset \mathbb{R}^n$ open $C^1(\Gamma^k(U)) = C^1$-smooth k-vector fields on U
$C^1(\Gamma_k(U)) = C^1$-smooth k-forms on U
[illegible]

Rmk: 0-forms = functions.

Suppose $\omega : E \to \mathrm{Alt}^0(T^*\mathbb{R}^n)$, that is, for every $p \in E$, $\omega(p) \in \mathrm{Alt}^0(T_p\mathbb{R}^n)$.

Since $\mathrm{Alt}^0(T_p\mathbb{R}^n) = L(\mathbb{R},\mathbb{R})$ by definition
and $L(\mathbb{R},\mathbb{R}) = \{ f : \mathbb{R}\to\mathbb{R} \text{ linear} \} = \{ f:\mathbb{R}\to\mathbb{R} : f(\ell) = \lambda \ell \}$
$\cong \mathbb{R}$,
we identify $\mathrm{Alt}^0(T_p\mathbb{R}^n) = \mathbb{R}$ and consider $\omega(p) \in \mathbb{R}$.
Thus $\omega : E \to \mathbb{R}$

Pull-back Let $U \subset \mathbb{R}^n$ open and $f : U \to \mathbb{R}^n$ C^1-smooth.
Let $\omega : E \to \Lambda^k T^*\mathbb{R}^n$ k-form s.t. $fU \subset E$.

Def: (pull-back) $f^*\omega : U \to \mathrm{Alt}^k(T^*\mathbb{R}^m)$

$$\underset{\substack{\uparrow\\ U}}{(f^*\omega)_p}\ \underset{\substack{\uparrow\\ T_p\mathbb{R}^m}}{(v_1,\dots,v_k)} = \omega_{f(p)}\big(Df(v_1),\dots,Df(v_k)\big).$$

Lemma:
$$f^*\Big(\sum_I u_I\, dx_{i_1}\wedge\dots\wedge dx_{i_k}\Big) = (u\circ f)\, f^*(dx_{i_1}\wedge\dots\wedge dx_{i_k})$$
$$= (u\circ f)\, f^*dx_{i_1}\wedge\dots\wedge f^*dx_{i_k}$$
$$= (u\circ f)\, df_{i_1}\wedge\dots\wedge df_{i_k}$$

Pf: Suffices to consider $f^*(u\, dx_{i_1}\wedge\dots\wedge dx_{i_k})$

$$f^*(u\, dx_{i_1}\wedge\dots\wedge dx_{i_k})_p(v_1,\dots,v_k) = (u\, dx_{i_1}\wedge\dots\wedge dx_{i_k})_{f(p)}\big(Df(v_1),\dots,Df(v_k)\big)$$

$$= u(f(p))\, dx_{i_1}\wedge\dots\wedge dx_{i_k}\big(Df(v_1),\dots,Df(v_k)\big)$$

$$= (u\circ f)(p)\ f^*(dx_{i_1}\wedge\dots\wedge dx_{i_k})_p(v_1,\dots,v_k)$$

$\varepsilon_{i_1}\wedge\dots\wedge\varepsilon_{i_k} = dx_{i_1}\wedge\dots\wedge\varepsilon_{i_k}$.

$$= (u\circ f)(p)\ (Df)_p(dx_{i_1}\wedge\dots\wedge dx_{i_k})(v_1,\dots,v_k)$$

$$= (u\circ f)(p)\ \Big(\underbrace{(Df)_p dx_{i_1}}_{= d(x_{i_1}\circ f) = df_{i_1}}\Big)\wedge\dots\wedge\big((Df)_p dx_{i_k}\big)(v_1,\dots,v_k)$$

$$= (u\circ f)(p)\ df_{i_1}\wedge\dots\wedge df_{i_k}\,(v_1,\dots,v_k). \qquad \square$$

Rmk: ① Given a C^1-smooth map $f: U \to \mathbb{R}^n$, $U \subset \mathbb{R}^m$ open, and $fU \subset E$, the map

$$f^*: \Gamma(E, Alt^k(T\mathbb{R}^n)) \longrightarrow \Gamma(U, Alt^k(T\mathbb{R}^m))$$

is linear: Let $\lambda \in \mathbb{R}$, $\omega, \tau \in \Gamma^k(E)$.

$$\begin{aligned}
f^*(\lambda\omega + \tau)_p(v_1 \ldots v_k) &= (\lambda\omega + \tau)_{f(p)}((Df)_p v_1, \ldots, (Df)_p v_k) \\
&= \lambda\, \omega_{f(p)}((Df)_p v_1 \ldots (Df)_p v_k) + \tau_{f(p)}((Df)_p v_1 \ldots (Df)_p v_k) \\
&= \lambda\, (f^*\omega)_p(v_1 \ldots v_k) + (f^*\tau)_p(v_1 \ldots v_k)
\end{aligned}$$

so

$$f^*(\lambda\omega + \tau) = \lambda f^*\omega + f^*\tau.$$

② Given $f: U \to V$ and $g: V \to \mathbb{R}^n$ C^1-smooth so that $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^l$ open and $fU \subset V$.

Then $(g \circ f)^* = f^* \circ g^*$

Indeed, let $\omega \in \Gamma^k(E)$ where $gV \subset E$.

Then

$$\begin{aligned}
(g\circ f)^*\omega_p(v_1 \ldots v_k) &= \omega_{g\circ f(p)}(D(g\circ f)_p v_1 \ldots D(g\circ f)_p v_k) \\
&= \omega_{g(f(p))}((Dg)_{f(p)} Df_p v_1 \ldots, (Dg)_{f(p)} Df_p v_k) \\
&= (g^*\omega)_{f(p)}((Df)_p v_1 \ldots (Df)_p v_k) \\
&= f^*(g^*\omega)_p(v_1, \ldots, v_k) \quad \square
\end{aligned}$$

Affine spaces

Let V be a vector space, $\dim V = n$.

Def: $P \subset V$ is a k-dim affine subspace of V if exists k-dim (linear) subspace $W \subset V$ and $p \in V$ so that

$$P = W + p = \{ w + p \in V : w \in W \}.$$

Lemma: Let $P = W + p$ where $W \subset V$ linear subspace and $p \in V$. Then

$$P = W + p'$$

for all $p' \in P$.

Pf: Let $p' \in P$. Then $p' = w' + p$ for some $w' \in W$. Thus $W + p' = W + w' + p = W + p$, since $w' \in W$. $\square$

Rmk: So W is uniquely defined.

Let V' be a finite dim vector space and $P' \subset V'$ affine subspace.

Def: A map $f : P \to P'$ is affine if

$$f(t x + (1-t) y) = t f(x) + (1-t) f(y)$$

for all $x, y \in P$ and $t \in \mathbb{R}$.

$f : P \to P'$ is an affine isomorphism if f is bijective.

Lemma: Let $P = W + p$ and $P' = W' + p'$ be affine subspaces of V and V' respectively and $f : P \to P'$ be an affine map. Then $g_f : W \to W'$, $g(w) = f(w+p) - f(p)$, is linear and does not depend on choice of $p \in P$.

Pf: Linearity: Let $v, w \in W$ and $\lambda \in \mathbb{R}$.

Then
$$\begin{aligned} g_f(\lambda v) &= f(\lambda v + p) - f(p) \\ &= f(\lambda(v+p) + (1-\lambda) p) - f(p) \\ &= \lambda f(v+p) + (1-\lambda) f(p) - f(p) \\ &= \lambda \big(f(v+p) - f(p)\big) = \lambda g_f(v) \end{aligned}$$

$$g_f(v+w) = 2\, g_f\left(\tfrac{1}{2}(v+w)\right) = 2\left[f\left(\tfrac{1}{2}v+\tfrac{1}{2}w+p\right) - f(p)\right]$$
$$= 2\left[f\left(\tfrac{1}{2}(v+p)+\tfrac{1}{2}(w+p)\right) - f(p)\right]$$
$$= 2\left[\tfrac{1}{2}f(v+p)+\tfrac{1}{2}f(w+p) - f(p)\right]$$
$$= 2\left[\tfrac{1}{2}g_f(v)+\tfrac{1}{2}g_f(w)\right] = g_f(v)+g_f(w)$$

Uniqueness: Let $p' \in P$. Then $p' = w' + p$ for $w' \in W$.

Then
$$f(v+p') - f(p') = f(v+w'+p) - f(w'+p)$$
$$= f(v+w'+p) - f(p) - f(w'+p) + f(p)$$
$$= g_f(v+w) - g_f(w) = g_f(v).$$

Thus the map $v \mapsto f(v+p') - f(p')$ is g_f.

$\square$

Def: We call the map $g_f : W \to W'$ as the linear map associated to $f: P \to P'$.

Let $P = W + p \subset V$ be an affine subspace.
Then $W^* = \{\text{linear maps } W \to \mathbb{R}\}$ is a (well-defined) vector space.

Def: $P^* := W^*$ (P^* is well-defined since W is unique.)

So elements of P^* are linear maps $W \to \mathbb{R}$.

Def: $\Lambda_k P := \Lambda_k W$ and $Alt^k(P) = Alt^k(W)$.

Let $f: P \to P'$ be an affine map $P = W + p$, $P' = W' + p'$.

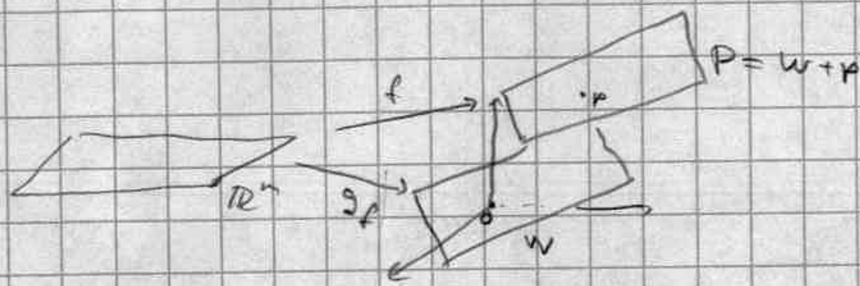
Def: $f_* : \Lambda_k P \to \Lambda_k P'$ is defined by $f_* := (g_f)_* : \Lambda_k W \to \Lambda_k W'$.

$f^* : Alt^k(P') \to Alt^k(P)$ is defined by $f^* := (g_f)^* : Alt^k(W') \to Alt^k(W)$

Def: $\xi \in \Lambda_k P$ is an orientation of P if $\xi \neq 0$ and $\dim P = k$

Special case $\mathbb{R}^m \to P$.

Let $P = W + p \subset \mathbb{R}^n$ be an affine subspace
and let $f : \mathbb{R}^m \to P$ be an affine map



Consider f as a map $\mathbb{R}^m \to \mathbb{R}^n$
Since $f(x) = g_f(x) + f(0)$ $\quad \left(g_f(v) = f(v+0) - f(0) \right)$
we have that for $x \in \mathbb{R}^m$ and $v \in \mathbb{R}^m$

$$\underbrace{(Df)_x((x,v))}_{\in T_x\mathbb{R}^m} = \underbrace{(f(x), g_f(v))}_{\in T_{f(x)}\mathbb{R}^n} \quad \text{since } Dg_f = g_f \text{ by linearity of } g_f$$

Thus $Df_x : T_x\mathbb{R}^m \to T_{f(x)}\mathbb{R}^n$ is the map
$$(x,v) \longmapsto (f(x), g_f(v))$$

In particular, $(Df_x)_* : \Lambda_k T_x\mathbb{R}^m \to \Lambda_k T_{f(x)}\mathbb{R}^n$
$$(x, \underset{\Lambda_k\mathbb{R}^m}{\xi}) \longmapsto (f(x), (g_f)_*\xi)$$

Rmk: Let $\xi : \mathbb{R}^m \to \Lambda_k\mathbb{R}^m$ be a k-vector field, $\xi(x) = (x, \xi_0(x))$
Then
$$(Df_x)_*(\xi(x)) = (Df_x)_*((x, \xi_0(x))) = (f(x), (g_f)_*\xi_0(x))$$

for every $x \in \mathbb{R}^m$.

If $f : \mathbb{R}^m \to P$ is an isomorphism, $y \mapsto (y, (g_f)_*\xi(f^{-1}(y)))$
is a k-vector field $P \to \Lambda_k\mathbb{R}^n$
We denote it by $f_*\xi$.
In this case $(f_*\xi)_y \in \{y\} \times W \subset T_y\mathbb{R}^n$.

Rmk: Suppose $(e_1', \dots, e_m')$ basis of W and $\xi = e_1' \wedge \dots \wedge e_m'$.
Then $f_*(e_1 \wedge \dots \wedge e_m) = J_f \, \xi$ and $J_f = \det(Df)_x$
for any $x \in \mathbb{R}^n$ when $\det Df$ is with respect to bases
$(e_1, \dots, e_m)$ and $(e_1', \dots, e_m')$.

General case: $P = W + p \subset \mathbb{R}^n$ affine subspace

Def: $T_x P = \{x\} \times W \subset T_x \mathbb{R}^n$ is the *tangent space* of P at $x \in P$.

Def: $TP := \bigcup_{x \in P} T_x P = \bigcup_{x \in P} \{x\} \times W \subset T\mathbb{R}^n$

is the tangent bundle of P.

Lemma: Let $P = W + p \subset \mathbb{R}^n$, $P' = W' + p' \subset \mathbb{R}^m$ be affine subspaces and $f : P \to P'$ an affine isomorphism.
Let $\xi : P \to \Lambda_k TP$ be a k-vectorfield. ($\Lambda_k TP \subset \Lambda_k T\mathbb{R}^n$)
Then $f_* \xi : P' \to \Lambda_k TP'$,
$$(f_* \xi)(y) = \left(y, (g_f)_* \left(\xi \circ f^{-1}(y)\right)\right)$$
is a well-defined k-vectorfield on P'.

Pf: Since f is a bijection $(f_* \xi)(y) \in \Lambda_k T\mathbb{R}^m$ is well-defined for every $y \in P'$.
Need to show that $(f_* \xi)(y) \in \Lambda_k TP'$.

Since $\xi(x) = (x, \xi_0(x))$, where $\xi_0(x) \in W$
and $g_f : W \to W'$, we have that $(g_f)_* \xi_0(x) \in W'$.
thus $f_* \xi(y) = (y, (g_f)_* \xi_0(f^{-1}(y))) \in \{y\} \times W'$ $\square$

Rmk: In particular, we observe that
$$f_*\left((x, \xi)\right) = \left(f(x), f_* \xi\right)$$
for orientation ξ of P. $\square$

A comment on Jacobians

Lemma: Let $f : P \to P'$ and $f' : P' \to P''$ be affine.
Then $g_{f' \circ f} = g_{f'} \circ g_f$

Pf: $P = W + p$, $P' = W' + p'$, $P'' = W'' + p''$.
and $g_{f' \circ f} : W \to W'$, $g_{f' \circ f}(v) = (f' \circ f)(v + p) - (f' \circ f)(p)$

Then
$$\begin{aligned} g_{f' \circ f}(v) &= f'(f(v+p)) - f'(f(p)) \\ &= f'(f(v+p) - f(p) + f(p)) - f'(f(p)) \\ &= g_{f'}(f(v+p) - f(p)) = g_{f'}(g_f(v)). \end{aligned}$$
$\square$

Cor: Let $f : P \to P'$ and $f' : P' \to P''$ affine.
Then $f'_* \circ f_* = (f' \circ f)_* : \Lambda_k P \to \Lambda_k P'$
and $f^* \circ (f')^* = (f' \circ f)^* : \mathrm{Alt}^k(P') \to \mathrm{Alt}^k(P)$
for all k

Pf:
$$f'_* \circ f_* = (g_{f'})_* \circ (g_f)_* = (g_{f'} \circ g_f)_* = (g_{f' \circ f})_* = (f' \circ f)_*$$
$$f^* \circ (f')^* = (g_f)^* \circ (g_{f'})^* = (g_{f'} \circ g_f)^* = (g_{f' \circ f})^* = (f' \circ f)^*$$
$\square$

Cor: Let (P, ξ), (P', ξ') and (P'', ξ'') be oriented affine subspaces of V, V', V'' of dimension k.
Let $f : P \to P'$ and $f' : P' \to P''$ be affine maps.
Then
$$J_{f' \circ f} = J_{f'} J_f$$

Pf:
$$\begin{aligned} (f' \circ f)_* \xi &= (g_{f'} \circ g_f)_* \xi = (g_{f'})_* (g_f)_* \xi = (g_{f'})_* (J_f \xi') \\ &= J_f (g_{f'})_* \xi' = J_f J_{f'} \xi'' \end{aligned}$$
$\square$

Integration of forms

$\mathbb{R}^n$: Let ω be a measurable n-form in $\mathbb{R}^n$
$\omega = u\, dx_1 \wedge \dots \wedge dx_n$ where $u : \mathbb{R}^n \to \mathbb{R}$ measurable

We say that ω is integrable if u is Lebesgue integrable and we define

$$\int_{\mathbb{R}^n} \omega := \int_{\mathbb{R}^n} u \, d\mathcal{L}^n$$

$\mathcal{L}^n$ — Lebesgue measure.

For $E \subset \mathbb{R}^n$ measurable, we set

$$\int_E \omega = \int_{\mathbb{R}^n} \chi_E \omega . \left(= \int_{\mathbb{R}^n} (\chi_E u)\, dx_1 \wedge \dots \wedge dx_n \right).$$

Lemma: Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be an affine map
and ω integrable n-form in $\mathbb{R}^n$, $\omega = u\, dx_1 \wedge \dots \wedge dx_n$

If $J_\varphi = 0$, then $\displaystyle\int_{\mathbb{R}^n} \varphi^* \omega = 0$.

If $J_\varphi \neq 0$ then $\displaystyle\int_{\mathbb{R}^n} \varphi^* \omega = (\operatorname{sign} J_\varphi) \int_{\mathbb{R}^n} \omega$.

Rmk: $\mathbb{R}^n$ is oriented with $e_1 \wedge \dots \wedge e_n$

Pf: $J_\varphi = 0 \Rightarrow g_\varphi \mathbb{R}^n$ has dim $< n \Rightarrow \mathcal{L}^n(\varphi \mathbb{R}^n) = 0$.

Thus, exercise

$$\int_{\mathbb{R}^n} \varphi^* \omega = \int_{\mathbb{R}^n} (u \circ \varphi)\, J_\varphi \, dx_1 \wedge \dots \wedge dx_n$$

$$= (\operatorname{sign} J_\varphi) \int_{\mathbb{R}^n} (u \circ \varphi)\, |J_\varphi| \, d\mathcal{L}^n$$

$$= (\operatorname{sign} J_\varphi) \int_{\varphi \mathbb{R}^n} u \, d\mathcal{L}^n = 0.$$

If $J_\varphi \neq 0 \Rightarrow g_\varphi : \mathbb{R}^n \to \mathbb{R}^n$ surj. $\Rightarrow \varphi\mathbb{R}^n = \mathbb{R}^n$.

Thus
$$\int_{\mathbb{R}^n} \varphi^*\omega = (\operatorname{sign} J_\varphi) \int_{\mathbb{R}^n} (u\circ\varphi)\,|J_\varphi|\, d\lambda^n$$

$$= (\operatorname{sign} J_\varphi) \int_{\mathbb{R}^n} u\, d\lambda^n = (\operatorname{sign} J_\varphi) \int_{\mathbb{R}^n} \omega .$$

$\square$

Lemma: Let (P,ξ) be an affine m-dimensional oriented subspace of $\mathbb{R}^n$. Let $\varphi, \psi : \mathbb{R}^m \to P$ be affine isomorphisms. Then

$$\int_{\mathbb{R}^m} \varphi^*\omega = \frac{\operatorname{sign} J_\psi}{\operatorname{sign} J_\varphi} \int_{\mathbb{R}^m} \psi^*\omega$$ if either integral is defined.

Pf: Suppose $\psi^*\omega$ is defined. Let $f : \mathbb{R}^m \to \mathbb{R}^m$, $f = \psi^{-1}\circ\varphi$. Then f is affine map. Since $\varphi = \psi\circ f$, $J_\varphi = J_\psi J_f$. Thus $\operatorname{sign} J_\varphi = (\operatorname{sign} J_\psi)(\operatorname{sign} J_f)$.

On the other hand,

$$(f^*\psi^*\omega)_x (v_1 \ldots v_m) = (\psi^*\omega)_{f(x)} \big((Df)_x v_1 \ldots (Df)_x v_m\big)$$

$$= \omega_{\psi(f(x))} \big((D\psi)_{f(x)} (Df)_x v_1, \ldots, (D\psi)_{f(x)} (Df)_x v_m\big)$$

$$= \omega_{(\psi\circ f)(x)} \big(D(\psi\circ f)_x v_1, \ldots, D(\psi\circ f)_x v_m\big)$$

$$= \omega_{\varphi(x)} \big((D\varphi)_x v_1 \ldots (D\varphi)_x v_m\big)$$

$$= (\varphi^*\omega)_x (v_1 \ldots v_m).$$

So $f^*\psi^*\omega = \varphi^*\omega$. Hence

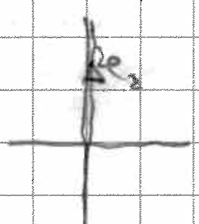
$$(\operatorname{sign} J_f) \int_{\mathbb{R}^m} \psi^*\omega = \int_{\mathbb{R}^m} f^*\psi^*\omega = \int_{\mathbb{R}^m} \varphi^*\omega \qquad \square$$

Def: Let (P, ξ) be an m-dimensional oriented affine subspace of $\mathbb{R}^n$, and $\omega : P \longrightarrow \mathrm{Alt}^m(T\mathbb{R}^n)$ an m-form. Then ω is *integrable* and

$$\int_P \omega = \int_{\mathbb{R}^m} \varphi^* \omega$$

if integral on the right exists for any affine orientation preserving isomorphism $\varphi : \mathbb{R}^m \longrightarrow P$.
(Here $\mathbb{R}^m$ is oriented with $e_1, \dots, e_m$.)

Example: $\varphi : \mathbb{R} \longrightarrow \mathbb{R}^2$, $t \longmapsto (0, t)$.
Thus $D\varphi = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



Let $\omega = u_1\, dx_1 + u_2\, dx_2$ be a continuous 1-form in $\mathbb{R}^2$
and $P = \{0\} \times \mathbb{R} = \varphi\mathbb{R}$.
Orient P with $2e_2$ and $\mathbb{R}$ with e_1.
Then $\varphi_* e_1 = \varphi e_1 = e_2$ so $J\varphi = 1/2$.

So
$$\int_P \omega = \int_{\mathbb{R}} \varphi^* \omega = \int_{\mathbb{R}} \varphi^*(u_1\, dx_1 + u_2\, dx_2)$$

$$= \int_{\mathbb{R}} u_1 \underbrace{d\varphi_1}_{=0} + u_2\, d\varphi_2$$

$$= \int_{\mathbb{R}} u_2\, dt = \int_{\mathbb{R}} u_2(t)\, d\lambda^1(t)$$

$$\left(= \int_{\mathbb{R}^1} \varphi^*(u_2\, dx_2) = \int_P u_2\, dx_2 \right)$$

$\Box$

Lemma: Let (P, ξ) be an oriented m-dim affine subspace of $\mathbb{R}^n$ and $\varphi : \mathbb{R}^m \to P$ an affine map. Suppose ω is an integrable m-form on P. Then

$$\int_{\mathbb{R}^m} \varphi^* \omega = \lambda \int_P \omega$$

where λ is the sign of J_φ if φ is an isom. and $\lambda = 0$ otherwise.

Pf. If φ is not an isom, then $\dim \operatorname{Im} \varphi < m$ so $\dim \operatorname{Im} (D\varphi)_x < m$ $((D\varphi)_x v_1, \ldots, (D\varphi)_x v_m)$ are lin dependent for all $v_1, \ldots, v_m$. So

$$\varphi^* \omega (v_1, \ldots, v_m) = 0.$$

Thus $\int_{\mathbb{R}^m} \varphi^* \omega = 0$ in this case.

If φ is an isom. and $J_\varphi > 0$ then we are done.

If φ is an isom and $J_\varphi < 0$. Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be the map $f(x_1, x_2 \ldots x_m) = (-x_1, x_2, \ldots x_m)$. Then $J_{\varphi \circ f} = J_f J_\varphi = (-1) J_\varphi > 0$.

Thus

$$\int_P \omega = \int_{\mathbb{R}^m} (\varphi \circ f)^* \omega = \int_{\mathbb{R}^m} f^* \varphi^* \omega$$

Let $\varphi^* \omega = u \, dx_1 \wedge \cdots \wedge dx_m$. Then

$$\int_{\mathbb{R}^m} f^* \varphi^* \omega = \int_{\mathbb{R}^m} (u \circ f) \, - dx_1 \wedge \cdots \wedge dx_m$$

$$= - \int_{\mathbb{R}^m} u(-x_1, x_2, \ldots x_m) \, d\mathcal{L}^m(x_1 \ldots x_m) \underset{\text{Leb. int.}}{=} - \int_{\mathbb{R}^m} \varphi^* \omega. \quad \square$$

Observation / Example:

Let $1 \le i_1 < i_2 < \dots < i_k \le n$ and let $\varphi : \mathbb{R}^k \to \mathbb{R}^n$ be the map

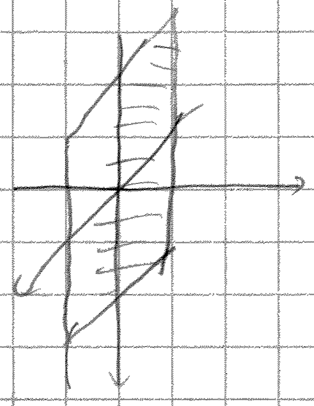
$$(x_1, \dots, x_k) \longmapsto \sum_{m=1}^{k} x_m e_{i_m}$$

Examples $n=3$, $k=2$, $i_1 = 1$, $i_2 = 3$: $(x_1, x_2) \mapsto (x_1, 0, x_2)$

Then $$\varphi^* dy_j = d(y_j \circ \varphi) = \begin{cases} dx_m, & \text{if } i_m = j \\ 0, & \text{otherwise} \end{cases}$$

$y = id : \mathbb{R}^n \to \mathbb{R}^n$
$y = (y_1, \dots, y_n)$

$$y_j \circ \varphi(x_1, \dots, x_k) = \begin{cases} x_m & \text{if } i_m = l \\ 0 & \text{otherwise} \end{cases}$$



$\varphi(\mathbb{R}^2) = P = \mathbb{R} \times \{0\} \times \mathbb{R}$

Thus
$$\varphi^*(\underbrace{dy_{j_1} \wedge \dots \wedge dy_{j_k}}_{dy_J}) = \begin{cases} \operatorname{sign} \sigma \; dx_1 \wedge \dots \wedge dx_k, & \sigma J = I \\ 0, & \text{otherwise.} \end{cases}$$

Let $\omega = \sum_J u_J \, dy_{j_1} \wedge \dots \wedge dy_{j_k}$ in $\mathbb{R}^n$.
Then orient $\varphi \mathbb{R}^k$ with $\varphi_*(e_1 \wedge \dots \wedge e_k) = e_{i_1} \wedge \dots \wedge e_{i_k}$!
Then φ is orientation preserving: $\mathbb{R}^k$ is oriented with $e_1 \wedge \dots \wedge e_k$.

So
$$\int_{\varphi \mathbb{R}^k} \omega = \int_{\mathbb{R}^k} \varphi^* \omega = \sum_J \int_{\mathbb{R}^k} (u_J \circ \varphi) \, \varphi^*(dy_{j_1} \wedge \dots \wedge dy_{j_k})$$

$$= \sum_{\substack{J \\ \sigma J = I}} \int_{\mathbb{R}^k} (u_J \circ \varphi) \operatorname{sign} \sigma \; dx_1 \wedge \dots \wedge dx_k.$$

Especially, if $(j_1, \dots, j_k)$ is <u>not</u> an permutation of $(i_1, \dots, i_k)$ then

$$\int_{\varphi \mathbb{R}^k} u \, dy_{j_1} \wedge \dots \wedge dy_{j_k} = 0$$

if it is then

$$\int_{\varphi \mathbb{R}^k} u \, dy_{j_1} \wedge \dots \wedge dy_{j_k} = \operatorname{sign} \sigma \int_{\varphi \mathbb{R}^k} u \, dy_{i_1} \wedge \dots \wedge dy_{i_k}.$$

$\square$