Exterior derivative II

Let ω be a k-form in $\mathbb{R}^n$.
We say that ω has compact support if

$$\operatorname{spt}\omega = \{p \in \mathbb{R}^n : \omega(p) \neq 0\} \text{ is compact}$$

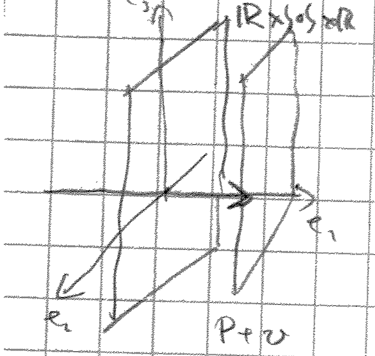
Let ω be a C^1-smooth compactly supported k-form.
Let $P \subset \mathbb{R}^n$ be k-dim. affine subspace of $\mathbb{R}^n$

Q: How the integrals $\int_P \omega$ vary when we move P?

Case study: ($k = n-1$)

Let $0 \leq l \leq n$ and $P = \mathbb{R}^{l-1} \times \{0\} \times \mathbb{R}^{n-l}$.
Orient P with $e_1 \wedge \dots \wedge \widehat{e_l} \wedge \dots \wedge e_n =: \xi$

$\uparrow$ e_l omitted.



Let $\omega = f\, dx_1 \wedge \dots \wedge \widehat{dx_l} \wedge \dots \wedge dx_n$

Let $v \in \mathbb{R}^n$ $v = w + v_l e_l$, $w \in P$ and $v_l \in \mathbb{R}$.

Observation: ① $P + v = P + v_l e_l$ oriented with ξ

② $\varphi : \mathbb{R}^{n-1} \longrightarrow P$, $(x_1, \dots, x_{n-1}) \longmapsto (x_1, \dots, \overset{l^{th}}{0}, \dots, x_{n-1})$

orientation preserving:

$$\varphi_*(e_1 \wedge \dots \wedge e_{n-1}) = \varphi_* e_1 \wedge \dots \wedge \varphi_* e_{n-1}$$
$$= e_1 \wedge \dots \wedge e_{l-1} \wedge e_{l+1} \wedge \dots \wedge e_n = \xi.$$

So

$$\int_P \omega = \int_{\mathbb{R}^{n-1}} \varphi^* \omega = \int_{\mathbb{R}^{n-1}} (f \circ \varphi)\, d\mathcal{L}^{n-1}$$

$$= \int_{\mathbb{R}^{n-1}} f(x_1, \dots, 0, \dots, x_{n-1})\, d\mathcal{L}^{n-1}(x_1, \dots, x_{n-1})$$

③ $\psi : \mathbb{R}^{n-1} \longrightarrow P + v$, $(x_1, \dots, x_{n-1}) \longmapsto (x_1, \dots, v_l, \dots, x_{n-1})$

orientation preserving: (linear map for ψ is φ)

$$\psi_*(e_1 \wedge \dots \wedge e_{n-1}) = \varphi_*(e_1 \wedge \dots \wedge e_{n-1}) = \xi.$$

So

$$\int_P \omega = \int_{\mathbb{R}^{n-1}} \psi^* \omega = \int_{\mathbb{R}^{n-1}} f(x_1, \dots, v_l, \dots, x_{n-1})\, d\mathcal{L}^{n-1}(x_1 \dots x_{n-1})$$

Then

$$\int_{P+v}\omega-\int_P\omega=\int_{\mathbb{R}^{n-1}}\Big(f(x_1,\dots,v_\ell,\dots,x_{n-1})-f(x_1,\dots,0,\dots,x_{n-1})\Big)\,d\lambda^{n-1}$$

$$=\int_{\mathbb{R}^{n-1}}\Big(\int_0^{v_\ell}\frac{\partial f}{\partial x_\ell}(x_1,\dots,\overset{\ell^{th}}{\overset{\downarrow}{t}},\dots,x_{n-1})\,dt\Big)\,d\lambda^{n-1}$$

Riemann

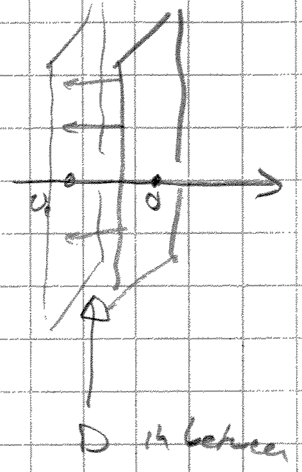
Lebesgue

S'pose $v_\ell>0$. Then, by Fubini,

$$\int_{\mathbb{R}^{n-1}}\int_0^{v_\ell}\frac{\partial f}{\partial x_\ell}(x_1,\dots,t,\dots,x_{n-1})\,dt\,d\lambda^{n-1}=\int_{\mathbb{R}^{\ell-1}\times[0,v_\ell]\times\mathbb{R}^{n-\ell}}\frac{\partial f}{\partial x_\ell}(x)\,d\lambda^n(x)$$

If $v_\ell<0$, then

$$\int_{\mathbb{R}^{n-1}}\int_0^{v_\ell}=-\int_{\mathbb{R}^{n-1}}\int_{v_\ell}^{0}=-\int_{\mathbb{R}^{\ell-1}\times[v_\ell,0]\times\mathbb{R}^{n-\ell}}\frac{\partial f}{\partial x_\ell}(x)\,d\lambda^n(x).$$



Denote $D=\begin{cases}\mathbb{R}^{\ell-1}\times[0,v_\ell]\times\mathbb{R}^{n-\ell}, & v_\ell>0\\ \mathbb{R}^{\ell-1}\times[v_\ell,0]\times\mathbb{R}^{n-\ell}, & v_\ell<0\end{cases}$

<u>Observation</u>: Orient $\mathbb{R}^n$ with $e_1\wedge\dots\wedge e_n$.

Then

$$\int_{P+v}\omega-\int_P\omega=\pm\int_D\frac{\partial f}{\partial x_\ell}\,d\lambda^n=\pm\int_D\frac{\partial f}{\partial x_\ell}\,dx_1\wedge\dots\wedge dx_n$$

depending on v

Recall:

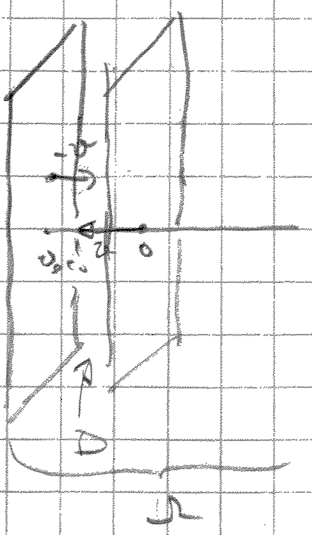
$\omega=f\,dx_1\wedge\dots\wedge\widehat{dx_\ell}\wedge\dots\wedge dx_n$

$$=\pm\int_D(-1)^{\ell-1}\frac{\partial f}{\partial x_\ell}\,dx_\ell\wedge dx_1\wedge\dots\wedge\widehat{dx_\ell}\wedge\dots\wedge dx_n$$

$df=\sum\frac{\partial f}{\partial x_k}dx_k$

$$=\pm(-1)^{\ell-1}\int_D df\wedge dx_1\wedge\dots\wedge\widehat{dx_\ell}\wedge\dots\wedge dx_n$$

Observation 2:



Orient $\mathbb{R}^n$ $v \wedge (e_1 \wedge \cdots \wedge \hat{e}_\ell \wedge \cdots \wedge e_n) = \mathcal{J}$
(Then P is pos. oriented in $\mathbb{R}^n$ with respect to the half space of $\mathbb{R}^n \setminus P$ containing $P+v$.)

Then
$$\mathcal{J} = (\operatorname{sign} v_\ell)\, e_\ell \wedge e_1 \wedge \cdots \wedge \hat{e}_\ell \wedge \cdots \wedge e_n$$
$$= (\operatorname{sign} v_\ell)\,(-1)^{\ell-1} e_1 \wedge \cdots \wedge e_n$$

So Jacobian $id : (\mathbb{R}^n, e_1 \wedge \cdots \wedge e_n) \to (\mathbb{R}^n, \mathcal{J})$
is $(\operatorname{sign} v_\ell)(-1)^{\ell-1}$

Thus

$$\int_{P+v} \omega - \int_P \omega = (\operatorname{sign} v_\ell)\,(-1)^{\ell-1} \int_{(\mathbb{R}^n, e_1 \wedge \cdots \wedge e_n)} \chi_D \, df \wedge dx_1 \wedge \cdots \wedge \widehat{dx_\ell} \wedge \cdots \wedge dx_n$$

$$= \int_{(\mathbb{R}^n, \mathcal{J})} \chi_D \, df \wedge dx_1 \wedge \cdots \wedge \widehat{dx_\ell} \wedge \cdots \wedge dx_n$$

$$= \int_D df \wedge dx_1 \wedge \cdots \wedge \widehat{dx_\ell} \wedge \cdots \wedge dx_n$$

when $\mathbb{R}^n$ is oriented with $\mathcal{J}$

Then

[illegible]

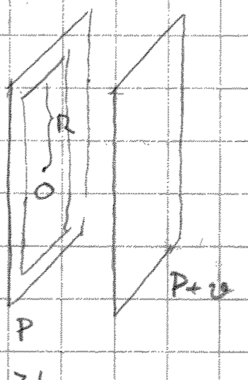
Case study 2: Let P and ξ be as in CS.1. Orient $\mathbb{R}^n$ with $\mathcal{J}$

Let $\omega = \sum f_k \, dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n$ compactly supported.

Then
$$\int_P \omega = \int_P f_\ell \, dx_1 \wedge \cdots \wedge \widehat{dx_\ell} \wedge \cdots \wedge dx_n$$
and
$$\int_{P+v} \omega = \int_{P+v} f_\ell \, dx_1 \wedge \cdots \wedge \widehat{dx_\ell} \wedge \cdots \wedge dx_n$$

On the other hand,
$$\int_D \sum_{k=1}^n df_k \wedge dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n$$
$$= \int_D \sum_{k=1}^n \frac{\partial f_k}{\partial x_k}(x) \, d\alpha^n(x) = \sum_{k=1}^n \int_D \frac{\partial f_k}{\partial x_k}(x) \, d\alpha^n(x).$$

Claim: $\displaystyle\sum_{k=1}^n \int_D \frac{\partial f_k}{\partial x_k} \, d\alpha^n = \int_D \frac{\partial f_\ell}{\partial x_\ell} \, d\alpha^n$



Pf: $\exists$ $R > 0$ so that support of ω is in $[-R,R]^n$.

Thus $\displaystyle\int_D \frac{\partial f_k}{\partial x_k} \, d\alpha^n = \int_{D \cap [-R,R]^n} \frac{\partial f_k}{\partial x_k} \, d\alpha$ for all $k \geq 1$.

Suppose $k \neq \ell$. Then, Fubini,
$$\int_{D \cap [-R,R]^n} \frac{\partial f_k}{\partial x_k} \, d\alpha^n = \int_{D_0} \int_{[-R,R]} \frac{\partial f_k}{\partial x_k}(x_1, \ldots \underset{\underset{k\text{th}}{\lrcorner}}{t}, \ldots x_n) \, dt \, d\alpha^{n-1}$$
$$\hookrightarrow D_0 = \{(x_1, \ldots, x_n) \in D : x_k = 0\}$$
$$= \int_{D_0} \Big(f_k(x_1 \ldots R \ldots x_n) - f_k(x_1 \ldots, -R, \ldots x_n) \Big) \, d\alpha^{n-1} = 0.$$

Conclusion: $\displaystyle\int \omega - \int_P \omega = \int_D \sum_k df_k \wedge dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n$

Def: Given a C^1-smooth k-form $\omega = \sum_{\substack{I=(i_1 \dots i_k)\\ 1 \le i_1 < \dots < i_k \le n}} u_I \, dx_{i_1} \wedge \dots \wedge dx_{i_k}$

we say that the $(k+1)$-form

$$d\omega := \sum_I du_I \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

is the <u>exterior</u> <u>derivative</u> of ω.

A fundamental property of the exterior derivative is the following.

Prop: Let $\omega \in C^1_0(\Gamma^{n-1}(\mathbb{R}^n))$ (a compactly supported C^1-smooth $(n-1)$-form in $\mathbb{R}^n$)
Then, for every $v \in \mathbb{R}^n_\ell$

$$\int_{P+v} \omega - \int_P \omega = \int_D d\omega$$

where $P = \mathbb{R}^{\ell-1} \times \{0\} \times \mathbb{R}^{n-\ell}$ and $P+v$ are oriented with $\xi = e_1 \wedge \dots \wedge \hat{e}_\ell \wedge \dots \wedge e_n$ and $\mathbb{R}^n$ with $\zeta = (\operatorname{sign} v_\ell)\, e_\ell \wedge e_1 \wedge \dots \wedge \hat{e}_\ell \wedge \dots \wedge e_n$
(Note: $\zeta = (\operatorname{sign} v_\ell)(-1)^\ell e_1 \wedge \dots \wedge e_n$) and $D = \{p + tv : p \in P,\ t \in [0,1]\}$.

Proposition gives the following "Baby Stokes"-theorem:

Thm: Let $\omega \in C^1_0(\Gamma^{n-1}(\mathbb{R}^n))$. Then

$$\int_{\mathbb{R}^{n-1} \times \{0\}} \omega = (-1)^n \int_{\mathbb{R}^{n-1} \times [0,\infty)} d\omega \qquad \text{(both with standard orientation)}$$

Pf: Take $R > 0$ so that $\operatorname{spt} \omega \subset [-R, R]^n$. Then
$R e_n \wedge e_1 \wedge \dots \wedge e_{n-1} = (-1)^{n-1} R\, e_1 \wedge \dots \wedge e_n$. Hence

$$-\int_{\mathbb{R}^{n-1} \times \{0\}} \omega = \int_{\mathbb{R}^{n-1} \times \{R\}} \omega - \int_{\mathbb{R}^{n-1} \times \{0\}} \omega = (-1)^{n-1} \int_{\mathbb{R}^{n-1} \times [0,R]} d\omega .$$

$\square$

Rmk: We denote the space of compactly supported C^k-smooth ℓ-forms in an open set $U \subset \mathbb{R}^n$ by $C^k_0(\Gamma^\ell(U))$ from now on.
Similarly $\Omega^\ell_0(U) = C^\infty_0(\Gamma^\ell(U))$.

<u>Lemma</u>: (definition

Let $\omega : E \to \mathrm{Alt}^k(T\mathbb{R}^n)$ be a C^1-smooth k-form.
Then

$$d\omega(v_1, \ldots, v_{k+1}) = \sum_{l=1}^{k+1} (-1)^l \, d\big(\omega(v_1 \ldots \hat{v}_l \ldots v_{k+1})\big)(v_l)$$

where $\omega(v_1, \ldots, \hat{v}_l \ldots v_{k+1})$ is the function
$p \longmapsto \omega_p(v_1, \ldots, v_{l-1}, v_{l+1}, \ldots, v_{k+1})$

Pf: Formulas are linear, may assume $\omega = f\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}$
Then

$$\sum_{l=1}^{k+1} (-1)^{l-1} d\big(\omega(v_1 \ldots \hat{v}_l \ldots v_{k+1})\big)(v_l)$$

$$= \sum_{l=1}^{k+1} (-1)^{l-1} \, d\Big(f \underbrace{dx_{i_1} \wedge \ldots \wedge dx_{i_k} (v_1 \ldots \hat{v}_l \ldots v_{k+1})}_{\text{constant} = c_I}\Big)(v_l)$$

$$= \sum_{l=1}^{k+1} (-1)^{l-1} df(v_l)\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}(v_1 \ldots \hat{v}_l \ldots v_{k+1})$$

$$= df \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} (v_1, \ldots, v_{k+1})$$

□

<u>Obs</u>: $d(dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = 0$
(directly from def.)

<u>Lemma</u>: (1) For all C^2-smooth ω, $d(d\omega) = 0$

(2) Suppose ω and τ are C^1 smooth k and l-forms, respectively
Then

$$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^k \omega \wedge d\tau$$

(3) Suppose $\varphi : \Omega \to \mathbb{R}^n$ be C^1-smooth and
$\omega : E \to \mathrm{Alt}^k(T\mathbb{R}^n)$ C^1-smooth k-form s.t. $f\Omega \subset E$
Then

$$\varphi^*(d\omega) = d(\varphi^*\omega)$$

<u>Pf</u>: (1) Recall: $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$.

Thus $d(df) = d\left(\sum_{l=1}^{n} \frac{\partial f}{\partial x_l} dx_l\right) = \sum_{l=1}^{n} \sum_{m=1}^{n} \frac{\partial f}{\partial x_l \partial x_m} dx_m \wedge dx_l$

$$= \sum_{l=1}^{n} \sum \qquad + \sum^{n} \sum \frac{\partial f}{\partial x_l \partial x_m} dx_m \wedge dx_l \qquad = 0$$

Claim (1) now follows if we prove (2). Indeed,

$$d(d\omega) = d(df \wedge dx_I) = \underbrace{d(df)}_{=0} \wedge dx_I + (-1)\, df \wedge \underbrace{d(dx_I)}_{=0}$$
$$= 0.$$

(2): Let $\omega = f\,dx_I$ and $\tau = g\,dx_J$.

Then

$$d(\omega \wedge \tau) = d(fg\, dx_I \wedge dx_J)$$
$$= \sum_{\ell=1}^{n} \frac{\partial(fg)}{\partial x_\ell}\, dx_\ell \wedge dx_I \wedge dx_J$$
$$= \sum_{\ell=1}^{n} \left(\frac{\partial f}{\partial x_\ell} g + f \frac{\partial g}{\partial x_\ell} \right) dx_\ell \wedge dx_I \wedge dx_J$$
$$= g \overbrace{\left(\sum_{\ell=1}^{n} \frac{\partial f}{\partial x_\ell} dx_\ell \wedge dx_I \right)}^{= df \wedge dx_I} \wedge dx_J + f \overbrace{\left(\sum_{\ell=1}^{n} \frac{\partial g}{\partial x_\ell} dx_\ell \right)} \wedge dx_I \wedge dx_J$$
$$= d\omega \wedge \tau + dg \wedge \omega \wedge dx_J$$
$$= d\omega \wedge \tau + (-1)^{1\cdot k}\, \omega \wedge dg \wedge dx_J$$
$$= d\omega \wedge \tau + (-1)^k\, \omega \wedge \tau$$

(3) $$\varphi^*(d\omega) = \varphi^*(df \wedge dx_I) = \varphi^* df \wedge \varphi^* dx_I$$

But $$d(f \circ \varphi)_p(v) \underset{\text{Chain rule}}{=} df_{\varphi(p)}(D\varphi_p(v)) \underset{\text{Def}}{=} (\varphi^* df)_p(v).$$

Thus

$$\varphi^*(d\omega) = \varphi^* df \wedge \varphi^* dx_I = d(f \circ \varphi) \wedge d\varphi_I$$
$$= d((f \circ \varphi)\, d\varphi_I) = d(\varphi^* \omega) \qquad \square$$

Lemma: Let $d_k : \Omega^k(U) \to \Omega^{k+1}(U)$ be linear operators, $k \geq 0$, so that

(1) $d_0 = d$

(2) $d_{k+1} \circ d_k = 0$ for all $k \geq 0$

(3) $d_{k+l}(\omega \wedge \tau) = d_k \omega \wedge \tau + (-1)^k \omega \wedge d_l \tau$ for all $k, l \geq 0$ and $\omega \in \Omega^k(U)$ and $\tau \in \Omega^l(U)$.

Then $d_k = d$ for all k.

Rmk: By lemma, exterior derivative is "unique".

Pf:

$$d_k(u\, dx_{i_1} \wedge \dots \wedge dx_{i_k}) \underset{(3)}{=} d_0 u \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} + u \wedge d_k(dx_{i_1} \wedge \dots \wedge dx_{i_k})$$

$$\underset{(1)\&(3)}{=} du \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} + u \wedge \left((d_1 dx_{i_1}) \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k} - dx_{i_1} \wedge d_{k-1}(dx_{i_2} \wedge \dots \wedge dx_{i_k}) \right)$$

$$\underset{(1)\&(2)}{=} d(u\, dx_{i_1} \wedge \dots \wedge dx_{i_k}) + u \wedge \left(0 - dx_{i_1} \wedge d_{k-1}(dx_{i_2} \wedge \dots \wedge dx_{i_k}) \right)$$

$$\underset{\text{Ind.}}{=} d(u\, dx_{i_1} \wedge \dots \wedge dx_{i_k}). \quad \square$$

Terminology: A form ω is closed if $d\omega = 0$.

A form ω is exact if exists τ so that $\omega = d\tau$.

Obs: (1) Every exact form is closed: $\omega = d\tau \Rightarrow d\omega = d^2\tau = 0$. $\square$

(2) An n-form in $U \subset \mathbb{R}^n$ is always closed.

Fantastic theorem

Thm: Let $\omega \in C_0^k(\Gamma^n(\mathbb{R}^n))$. Then

$$\int_{\mathbb{R}^n} \omega = 0$$ if and only if ω is exact and $\omega = d\tau$ where $\tau \in C_0^k(\Gamma^{n-1}(\mathbb{R}^n))$.

Pf: $\Leftarrow$ Exercise

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$\Rightarrow$ Suppose $\int_{\mathbb{R}^n} \omega = 0$ and ω is compactly supported C^k-smooth

Then $\omega = f\, dx_1 \wedge \dots \wedge dx_n$. Let $R > 0$ s.t. $\operatorname{spt} f \subset (-R, R)^n$.

Induction on n:

<u>$n=1$</u>: Set $F(x) = \int_{-\infty}^{x} f(t)\, d\mathcal{L}^1(t)$. Then $dF = f\, dx$

Moreover, if $x < -R$, then $F(x) = 0$
If $x > R$ then

$$F(x) = \int_{-\infty}^{x} f(t)\, d\mathcal{L}^1(t) = \int_{-\infty}^{\infty} f(t)\, d\mathcal{L}^1(t) = \int_{\mathbb{R}} f\, dx = \int_{\mathbb{R}} \omega = 0$$

Thus $\operatorname{spt} F \subset (-R, R)$, and F is compactly supported.

Suppose now that $n \geq 2$ and claim holds for $n-1$.

<u>Idea 1</u>: Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}$, $g(x_1, x_2, \dots, x_n) = \int_{-\infty}^{x_n} f(x_1 \dots x_{n-1}, t)\, d\mathcal{L}^1(t)$.

$$\text{Then } d\left((-1)^{n-1} g\, dx_1 \wedge \dots \wedge dx_{n-1}\right) = (-1)^{n-1} dg \wedge dx_1 \wedge \dots \wedge dx_{n-1}$$
$$= \frac{\partial g}{\partial x_n} dx_1 \wedge \dots \wedge dx_n = f\, dx_1 \wedge \dots \wedge dx_n$$

Then $\omega = d\left((-1)^{n-1} g\, dx_1 \wedge \dots \wedge dx_n\right)$

But is $(-1)^{n-1} g\, dx_1 \wedge \dots \wedge dx_n$ compactly supported?

<u>No!</u> The function $(x_1, \dots, x_{n-1}) \longmapsto g(x_1, \dots, x_{n-1}, R)$
is not a zero function in general!
Since $f(x_1, \dots, x_{n-1}, t) = 0$ for all $t > R$
we have $g(x_1, \dots, x_{n-1}, t) = g(x_1, \dots, x_{n-1}, R)$ for $t > R$!
Thus g need not have a compact support.

Second approach:

Suppose that the claim holds in $\mathbb{R}^k$ for $k \le n-1$.

Define $g : \mathbb{R}^{n-1} \to \mathbb{R}$ by $g(x_1 \dots x_{n-1}) = \int_{\mathbb{R}} f(x_1 \dots x_{n-1}, t)\, dt$.

Then
$$\int_{\mathbb{R}^{n-1}} g\, dx_1 \wedge \dots \wedge dx_{n-1} = \int_{\mathbb{R}^{n-1}} g(x_1 \dots x_{n-1})\, d\mathcal{L}^{n-1}$$
$$= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} f(x_1 \dots x_{n-1}, t)\, dt\, d\mathcal{L}^{n-1}$$
$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}^n} f(x_1 \dots x_n)\, d\mathcal{L}^n(x_1 \dots x_n) = \int_{\mathbb{R}^n} \omega = 0.$$

Thus, by induction assumption, there exists an $C_0^k(\Gamma^{n-2}(\mathbb{R}^{n-1}))$-form
$$\tilde{\tau} = \sum_{k=1}^{n-1} (-1)^{k-1} h_k\, dx_1 \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_{n-1}$$
in $\mathbb{R}^{n-1}$ so that $g\, dx_1 \wedge \dots \wedge dx_{n-1} = d\tilde{\tau}$.

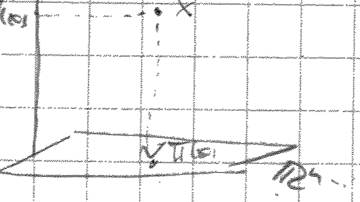
Since
$$d\tilde{\tau} = \sum_{k=1}^{n-1} (-1)^{k-1} dh_k \wedge dx_1 \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_{n-1}$$
$$= \sum_{k=1}^{n-1} (-1)^{k-1} \frac{\partial h_k}{\partial x_k}\, dx_k \wedge dx_1 \wedge \dots \wedge \widehat{dx_k} \wedge \dots \wedge dx_{n-1}$$
$$= \left(\sum_{k=1}^{n-1} \frac{\partial h_k}{\partial x_k} \right) dx_1 \wedge \dots \wedge dx_{n-1},$$
we have that $g = \sum_{k=1}^{n-1} \frac{\partial h_k}{\partial x_k}$.

Back to $\omega = f\, dx_1 \wedge \dots \wedge dx_n$:

Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ $(x_1 \dots x_n) \mapsto (x_1 \dots x_{n-1})$

$\pi' : \mathbb{R}^n \to \mathbb{R}$ $(x_1 \dots x_n) \mapsto x_n$

be projections.



Auxiliary forms

Fix $u \in C_0^\infty(-R, R)$ $\int_{\mathbb{R}} u\, dx' = 1$.

Set $\beta = (\pi')^*(u\, dx) = (u \circ \pi')\, dx_n \in C^\infty(\Gamma^1(\mathbb{R}^n))$

Since $d(u\, dx) = du \wedge dx = u'\, dx \wedge dx = 0$,

we have $d\beta = d(\pi')^*(u\, dx) = (\pi')^* d(u\, dx) = 0$.

i.e. β is closed.

Set $\alpha = \pi^* \tilde{\tau} \in C^1(\Gamma^{n-2}(\mathbb{R}^n))$.

Then $d\alpha = \pi^* d\tilde{\tau} = \pi^*(g\, dx_1 \wedge \dots \wedge dx_{n-1}) = (g \circ \pi)\, dx_1 \wedge \dots \wedge dx_{n-1}$

Thus $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{n-2} \alpha \wedge d\beta = d\alpha \wedge \beta$.
$= (g \circ \pi)(u \circ \pi')\, dx_1 \wedge \dots \wedge dx_n$.

Note that $d(\alpha \wedge \beta)$ has compact support

We set $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(x_1 \dots x_n) = \int_{-\infty}^{x_n} f(x_1 \dots x_{n-1}, t)\, dt - \underbrace{\Big(\overbrace{\int_{-\infty}^{x_n} u(t)\, dt}^{=1 \text{ for } x_n \geq R}\Big)\, g(x_1 \dots x_{n-1})}_{= g(x_1 \dots x_{n-1}) = \int_{-\infty}^{\infty} f(x_1 \dots x_{n-1}, t)\, dt \text{ for } x_n \geq R}.$$

Then
$$\frac{\partial F}{\partial x_n}(x) = f(x_1 \dots x_n) - u(x_n)\, g(x_1 \dots x_{n-1})$$
$$= f(x) - (u \circ \pi')(x)\,(g \circ \pi)(x)$$

and

$$d\big((-1)^n F\, dx_1 \wedge \dots \wedge dx_{n-1}\big) = \frac{\partial F}{\partial x_n}\, dx_1 \wedge \dots \wedge dx_n = \big(f - (u \circ \pi')(g \circ \pi)\big)\, dx_1 \wedge \dots \wedge dx_n$$
$$= \omega - d(\alpha \wedge \beta)$$

Then $\omega = d\big((-1)^n F\, dx_1 \wedge \dots \wedge dx_{n-1}\big) + d(\alpha \wedge \beta)$

Suffices to show that F has compact support.

If $(x_1 \dots x_{n-1}) \notin [-R, R]^{n-1}$ then $f(x_1 \dots x_{n-1}, t) = 0$ for all t.

Thus $g(x_1 \dots x_{n-1}) = 0$ and $F(x_1 \dots x_n) = 0$.

If $(x_1,\dots,x_{n-1}) \in [-R,R]^{n}$ and $x_n > R$ then

$$f_n(x_1,\dots,x_n) = \int_{-\infty}^{x_n} f(x_1 \dots x_{n-1}, t)\,dt - \Big(\underbrace{\int_{-\infty}^{x_n} u(t)\,dt}_{=1}\Big)\, g(x_1 \dots x_{n-1})$$

$$= \underbrace{\int_{-\infty}^{\infty} f(x_1 \dots x_{n-1}, t)\,dt}_{= g(x_1 \dots x_{n-1})} - g(x_1 \dots x_{n-1}) = 0.$$

Similarly for $x_n < R$.

Smoothness:

$$\begin{tikzcd} f \in C^k(\mathbb{R}^n) \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & g \in C^k(\mathbb{R}^{n-1}) \arrow[d, Rightarrow] \\ & f_n \in C^k(\mathbb{R}^n) \end{tikzcd}$$

□

Corollary: Kernel of the integral operator

$$\int_{\mathbb{R}^n} : \Omega^n_0(\mathbb{R}^n) \longrightarrow \mathbb{R}$$

is $d\,\Omega^{n-1}_0(\mathbb{R}^n)$ (= exact compactly supported forms)

In particular,

$$\begin{tikzcd} \Omega^n_0(\mathbb{R}^n) \arrow[r, "\int_{\mathbb{R}^n}"] \arrow[dr] & \mathbb{R} \\ & \Omega^n_0(\mathbb{R}^n) / d\Omega^{n-1}_0(\mathbb{R}^n) \arrow[u, "\cong"'] \end{tikzcd}$$

Rmk: The one dimensional space

$$\Omega^n_0(\mathbb{R}^n) \Big/ d\,\Omega^{n-1}_0(\mathbb{R}^n) \cong \mathbb{R}$$

is called the compactly supported nth de Rham cohomology of $\mathbb{R}^n$.

Obs: In the proof we get $\Omega^n(\mathbb{R}^n) = d\,\Omega^{n-1}(\mathbb{R}^n)$

thus $\Omega^n(\mathbb{R}^n) \Big/ d\,\Omega^{n-1}(\mathbb{R}^n) = \{0\}$