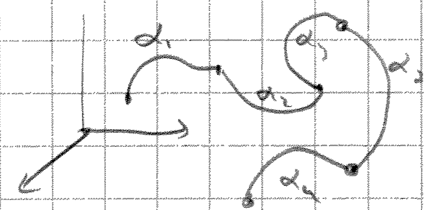
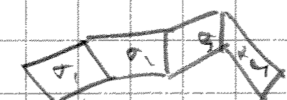
Chain complexes

<u>Motivation</u> Suppose $\omega \in C^0(\Gamma^1(\mathbb{R}^n))$
and $\alpha_i : [0,1] \longrightarrow \mathbb{R}^n$ C^1-paths so that



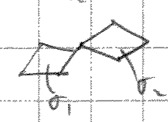
Then $$\int_{\alpha_4 * \alpha_3 * \alpha_2 * \alpha_1} \omega = \sum_{i=1}^{4} \int_{\alpha_i} \omega .$$

Similarly, if $\tau \in C^0(\Gamma^2(\mathbb{R}^n))$
and $\sigma_i : [0,1]^2 \longrightarrow \mathbb{R}^n$



we have $$\sum_{i=1}^{4} \int_{\sigma_i} \tau := \sum_{i=1}^{4} \int_{[0,1]^2} \sigma_i^* \tau .$$

<u>Q:</u> What is the counterpart for $\alpha_4 * \cdots * \alpha_1$?



<u>Side remark:</u> Suppose  then

$$\int_{\alpha_2 * \alpha_1} \omega = \int_{\alpha_1 * \alpha_2} \omega$$

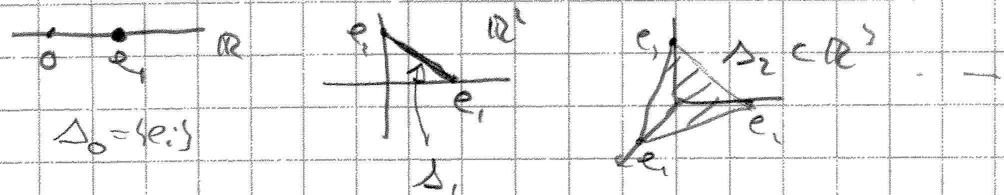
$\Rightarrow$ order of concatenation does not really matter

$$\Rightarrow \text{Idea: } \int_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4} \omega = \sum_{i=1}^{4} \int_{\alpha_i} \omega$$

Singular simplices.

Def: The standard k-simplex is

$$\Delta_k = \left\{ \sum_{i=1}^{k+1} t_i e_i \;:\; 0 \le t_i \le 1,\ \sum_{i=1}^{k+1} t_i = 1 \right\} \subset \mathbb{R}^{k+1}$$



Def: A singular k-simplex (in top space X) is a continuous map
$$\sigma : \Delta_k \longrightarrow X.$$

Def: An affine k-simplex in $U \subset \mathbb{R}^n$ is
$$\sigma : \Delta_k \longrightarrow U$$
s.t. σ is affine $\left(\sigma(tx + (1-t)y) = t\sigma(x) + (1-t)\sigma(y) \right)$

Def: A C^1-smooth k-simplex (in $U \subset \mathbb{R}^n$) is
$$\sigma : \Delta_k \longrightarrow U$$
s.t. σ is C^1-smooth.

Notation: $\sigma : \Delta_k \longrightarrow \mathbb{R}^n$ affine is also denoted
$[v_1, \dots, v_k] : \Delta_k \longrightarrow \mathbb{R}^n$ where
$v_i = \sigma(e_i)$. Then $[v_1, \dots v_k]\left(\sum_{i=1}^{k+1} t_i e_i\right) = \sum_{i=1}^{k+1} t_i v_i$

Notation: Denote $S_k(X) = \{ \sigma : \Delta_k \longrightarrow X : \sigma \text{ cont.} \}$ $k \ge 0$

Convention: $S_{-1}(X) = \emptyset = S_{-k}(X)$ $\forall\, k > 0$

Chains:

Def: The space of <u>k-chains in X</u> is

$$C_k(X) = \mathcal{F}_{\mathbb{Z}}(S_k(X)) = \operatorname{span}_{\mathbb{Z}}\{ \chi_{\{\sigma\}} : \sigma \in S_k(X)\}$$

(the free $\mathbb{Z}$-module over $S_k(X)$)

Rmks:
- Elements of $C_k(X)$ are finite sums
$$\sum_{i=1}^{m} a_i \chi_{\{\sigma_i\}} \quad \text{where } a_i \in \mathbb{Z} \text{ and } \sigma_i \in S_k(X).$$
- $C_k(X)$ is an abelian group in $\mathbb{R}^{S_k(X)}$
- $C_{-1}(X) = \{0\} = C_{-k}(X) \quad \forall k > 0$.
- We denote (as everybody)
$$\sum_{i=1}^{m} a_i \chi_{\{\sigma_i\}} =: \sum_{i=1}^{m} a_i \sigma_i$$
- Note: $\sigma_i \in S_k(\mathbb{R}^n)$: $\underset{\substack{\uparrow \\ \text{in } C_k(\mathbb{R}^n)}}{\sigma_0 + \sigma_1} = \chi_{\{\sigma_0\}} + \chi_{\{\sigma_1\}} \neq \chi_{\{\underset{\substack{\uparrow \\ \text{in } \mathbb{R}^n}}{\sigma_0 + \sigma_1}\}}$

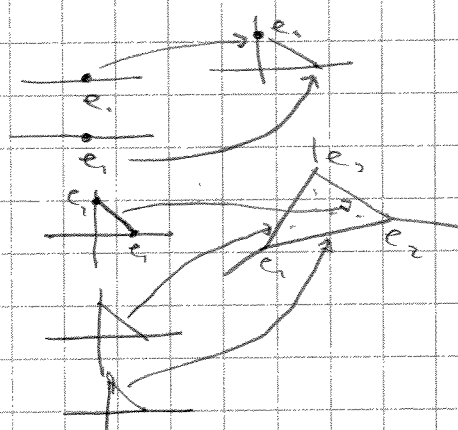
Chain complex

The boundary operator

Def: We call the map $[e_1,\dots,\hat{e}_i,\dots,e_{k+1}] : \Delta_{k-1} \longrightarrow \Delta_k$ as the ith face of Δ_k.

Ex: $k=1$: $i=1$: $[e_2]$
$i=2$: $[e_1]$

$k=2$: $i=1$: $[e_2, e_3]$
$i=2$: $[e_1, e_3]$
$i=3$: $[e_1, e_2]$



Def: The boundary operator

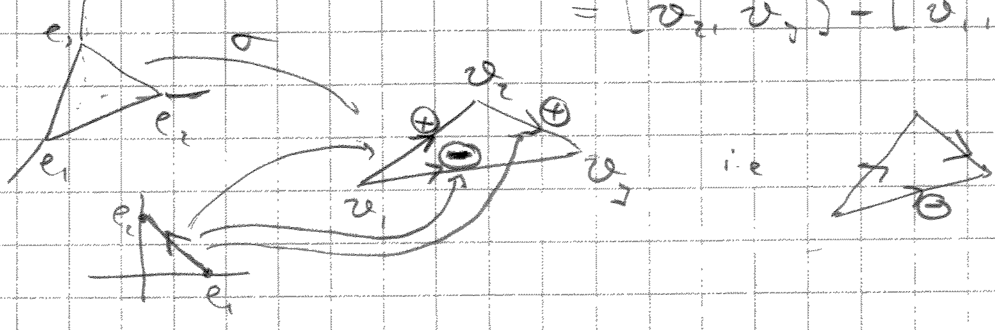
$$\partial : C_k(X) \longrightarrow C_{k-1}(X)$$

is the homomorphism

$$\partial(\sigma) = \sum_{i=1}^{k+1} (-1)^{i-1}\, \sigma \circ [e_1 \dots \hat{e}_i \dots e_{k+1}]$$

$\sigma = \sum a_i \sigma_i \Rightarrow$
$\sigma \circ f = \sum_i a_i (\sigma_i \circ f)$

Ex: $k=2$:
$$\begin{aligned}\partial[v_1, v_2, v_3] &= [v_1, v_2, v_3]\circ[e_2, e_3] \\ &\quad - [v_1, v_2, v_3]\circ[e_1, e_3] \\ &\quad + [v_1, v_2, v_3]\circ[e_1, e_2] \\ &= [v_2, v_3] - [v_1, v_3] + [v_1, v_2]\end{aligned}$$



Lemma: Let $f : X \to Y$ continuous map. Then
$$f_* : C_k(X) \longrightarrow C_k(Y)$$
$$f_*\left(\sum_i a_i \sigma_i\right) = \sum_i a_i\, f \circ \sigma_i$$
is a homomorphism.

Pf: Note that $\sigma_i = \chi_{\{\sigma_i\}}$ and $f\circ\sigma_i = \chi_{\{f\circ\sigma_i\}} \in S_k(Y)$ $\square$

Lemma: $\partial \circ f_\# = f_\# \circ \partial$

Pf: Suffices to show $\partial f_\# \sigma = f_\# \partial \sigma$ for $\sigma \in S_k(X)$.
Since $f_\# \sigma = f \circ \sigma$, we have that

$$\partial f_\# \sigma = \partial (f \circ \sigma) = \sum_{i=1}^{k+1} (-1)^{i-1} (f \circ \sigma) \circ [e_1 \dots \hat{e}_i \dots e_{k+1}]$$
$$= \sum_{i=1}^{k+1} (-1)^{i-1} f \circ (\sigma \circ [e_1 \dots \hat{e}_i \dots e_{k+1}])$$
$$= f_\# \left(\sum_{i=1}^{k+1} (-1)^{i-1} \sigma \circ [e_1 \dots \hat{e}_i \dots e_{k+1}] \right)$$
$$= f_\# \partial \sigma. \quad \square$$

Lemma: $\partial \circ \partial = 0$

Pf: Since ∂ is a homomorphism, suffices to show $\partial^2 \sigma = 0$ for $\sigma \in S_k(X)$. Since $\sigma = \sigma_\# [e_1 \dots e_{k+1}]$,
$\partial^2 \sigma = \sigma_\# \, \partial^2 [e_1 \dots e_{k+1}]$
For $k = 0, 1$, $\partial^2 [e_1, \dots, e_{k+1}] = 0$. (Thus $\partial^2 \sigma = 0$ for $\sigma \in S_k(X)$ $k = 0, 1$)
Suppose that holds for $k-1$. Then

$$\partial^2 [e_1 \dots e_{k+1}] = \partial \left(\sum_{i=1}^{k+1} (-1)^{i-1} \overbrace{[e_1 \dots e_{k+1}]}^{= id} \circ [e_1 \dots \hat{e}_i \dots e_{k+1}] \right)$$
$$= \partial \left(\sum_{i=1}^{k+1} (-1)^{i-1} [e_1 \dots \hat{e}_i \dots e_{k+1}] \right)$$
$$= \partial \left(\sum_{i=1}^{k+1} (-1)^{i-1} [e_1, \dots, e_{i-1}, e_{i+1}, \dots e_{k+1}] \right)$$
$$= \sum_{i=1}^{k+1} (-1)^{i-1} \left(\sum_{j=1}^{i-1} (-1)^{j-1} [e_1, \dots \hat{e}_j, \dots \hat{e}_i \dots e_{k+1}] + \sum_{j=i}^{k} (-1)^{j-1} [e_1, e_{\dots}, \hat{e}_i, \dots, \hat{e}_{j+1}, \dots e_{k+1}] \right)$$
$$= \underbrace{\sum_{j<i} (-1)^{i+j} [e_1, \dots \hat{e}_j, \dots, \hat{e}_i, \dots e_{k+1}] + \sum_{j \geq i} (-1)^{i+j} [e_1, \dots, \hat{e}_i, \dots \hat{e}_{j+1}, \dots e_{k+1}]}_{= \sum_{j>i} (-1)^{i+j-1} [e_1 \dots \hat{e}_i \dots \hat{e}_j \dots e_{k+1}]} = 0 \quad \square$$

Integration of forms over chains

Q: How to define $\displaystyle\int_\sigma \omega$ when ω is a continuous k-form in $U \subset \mathbb{R}^n$ and $\sigma : \Delta_k \to U$ a k-simplex ?
(or more generally a chain)

Approach 1: If there exists a neighborhood W of Δ_k and a C^1-extension of σ to $\tilde{\sigma} : V \to U$ then $\tilde{\sigma}^* \omega$ is a continuous k-form in V.
Then

$$\int_{\Delta_k} \sigma^* \omega = \int_{(P_k, \xi_k)} \chi_{\Delta_k} \tilde{\sigma}^* \omega$$

is well defined, where $P_k = e_1 + \operatorname{span}\{e_2 - e_1, \dots, e_{k+1} - e_1\}$
$= e_1 + W_k$
is the affine k-plane containing Δ_k
and ξ_k is the orientation $\xi_k = (e_2 - e_1) \wedge \dots \wedge (e_{k+1} - e_1)$.

Approach 2: Let $\Delta_k^0 = \{(x_1 \dots x_k) \in \mathbb{R}^k : 0 \le x_j \le 1, \sum x_i \le 1\}$
Orient $\mathbb{R}^k$ with $e_1 \wedge \dots \wedge e_k$
Take $\rho_k : \Delta_k^0 \to \Delta_k$ the affine map $\rho_k(0) = e_1 \quad \rho_k(e_i) = e_{i+1}$

i.e. $\rho_k(x_1 \dots x_k) = e_1 + \sum_{j=1}^k x_j (e_{j+1} - e_1)$
Then the linear part of ρ_k is $(\rho_k)_0(x) = \rho_k(x) - \rho_k(0) \;\; \left[= \sum_{j=1}^k x_j (e_{j+1} - e_1)\right]$
Thus $(\rho_k)_* e_1 \wedge \dots \wedge e_k = (e_2 - e_1) \wedge \dots \wedge (e_{k+1} - e_1)$
Hence ρ_k is orientation preserving

Let $\omega \in C^0(\Gamma^k(U))$ and $\sigma \in S_k(U)$
so that $\sigma \circ \rho_k : \Delta_k^0 \to U$ is continuous and C^1-smooth in the interior of $\Delta_k^0 \subset \mathbb{R}^k$
We define

$$\int_\sigma \omega = \int_{\Delta_k^0} (\sigma \circ \rho_k)^* \omega .$$

Rmk: Under assumptions of (1), we have

$$\int_{P_k} \chi_{\Delta_k}\, \sigma^*\omega = \int_{\Delta_k^\circ} \rho_k^*(\sigma^*\omega) = \int_{\Delta_k^\circ} (\sigma\circ\rho_k)^*\omega .$$

Conclusion 1: For the purpose of integration, consider maps $\Delta_k^\circ \to U$ instead of $\Delta_k \to U$.

To formulate Stokes's thm, we need chains and boundary operators for maps $\Delta_k^\circ \to U$.

What theory can be developed:

$$\tilde S_k(U) = \{\sigma : \Delta_k^\circ \to U : \sigma \text{ cont}\}$$
$$\tilde C_k(U) = \left\{\sum_{i=1}^m a_i\tilde\sigma_i : a_i\in\mathbb{Z},\ \tilde\sigma_i \in \tilde S_k(U)\right\} \subset \mathbb{R}^{\tilde S_k(U)}$$

Denote $\tilde\rho_k : C_k(U) \to \tilde C_k(U)$

$$\tilde\rho_k\left(\sum_{i=1}^m a_i\sigma_i\right) = \sum_{i=1}^m a_i(\sigma_i\circ\rho_k).$$

and $\tilde\rho_k^{-1} : \tilde C_k(U) \to C_k(U)$

$$\tilde\rho_k^{-1}\left(\sum_{i=1}^m a_i\tilde\sigma_i\right) = \sum_{i=1}^m a_i\,\tilde\sigma_i\circ\rho_k^{-1}$$

Define $\tilde\partial_k : \tilde C_k(U) \to \tilde C_{k-1}(U)$ by

$$\tilde\partial_k\tilde\sigma = \tilde\rho_{k-1}\partial_k\tilde\rho_k^{-1}\tilde\sigma.$$

Then

$$\tilde\partial_{k-1}\circ\tilde\partial_k = \tilde\rho_{k-2}\,\partial_{k-1}\,\cancel{\tilde\rho_{k-1}^{-1}\,\tilde\rho_{k-1}}\,\partial_k\,\tilde\rho_k^{-1} = \tilde\rho_{k-2}\,\partial^2\,\tilde\rho_k^{-1} = 0.$$

We define
$$\int_{\tilde\sigma}\omega = \int_{\Delta_k^\circ}\tilde\sigma^*\omega \qquad (U)$$

for $\omega\in C^0(\Pi^k(U))$, $\tilde\sigma\in\tilde S_k(U)$ C^1-smooth.

We do not pursue this direction.

Another Stokes' Theorem

Thm: Suppose $\omega \in C^1(\Gamma^{k-1}(U))$ and $\sigma \in S_k(U)$ has a C^1-extension to a neighborhood of Δ_k. Then

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega$$

Pf: Denote affine extension of $\rho_k : \Delta_k^0 \to \Delta_k$ to a neighborhood of Δ_k^0 also by ρ_k.

Step 1:
$$\int_\sigma d\omega = \int_{\Delta_k} \sigma^* d\omega = \int_{\Delta_k^0} \rho_k^* \sigma^* d\omega = \int_{\Delta_k^0} d\, \rho_k^* \sigma^* \omega$$

$$\int_{\partial\sigma} \omega = \sum_{i=1}^{k+1} (-1)^{i-1} \int_{\sigma \circ [e_1 \dots \hat{e}_i \dots e_{k+1}]} \omega = \sum_{i=1}^{k+1} (-1)^{i-1} \int_{\Delta_{k-1}^0} \rho_{k-1}^* [e_1 \dots \hat{e}_i \dots e_{k+1}]^* \sigma^* \omega$$

Step 2: We observe that
$$\rho_{k-1}^* [e_1 \dots \hat{e}_i \dots e_{k+1}]^* \sigma^* \omega = \rho_{k-1}^* [e_1 \dots \hat{e}_i \dots e_{k+1}]^* (\rho_k^{-1})^* \rho_k^* \sigma^* \omega$$

Then
$$\sum_{i=1}^{k+1} (-1)^{i-1} \int_{\Delta_{k-1}^0} \rho_{k-1}^* [e_1 \dots \hat{e}_i \dots e_{k+1}]^* \sigma^* \omega$$
$$= \sum_{i=1}^{k+1} (-1)^{i-1} \int_{\Delta_{k-1}^0} \rho_{k-1}^* [e_1 \dots \hat{e}_i \dots e_{k+1}]^* (\rho_k^{-1})^* \rho_k^* \sigma^* \omega$$

We denote $F_i : \Delta_{k-1}^0 \to \Delta_k^0$ the map

$$F_i = \rho_k^{-1} \circ [e_1 \dots \hat{e}_i \dots e_{k+1}] \circ \rho_{k-1}$$

Then
$$\int_{\partial\sigma} \omega = \sum_{i=1}^{k+1} (-1)^{i-1} \int_{\Delta_{k-1}^0} F_i^* \rho_k^* \sigma^* \omega$$

Moreover, $F_i(x_1 \dots x_{k-1}) = \rho_k^{-1} [e_1 \dots \hat{e}_i \dots e_{k+1}] \left(e_1 + \sum_{j=1}^{k-1} x_j (e_{j+1} - e_1)\right)$

for $i > 1$
$$= \rho_k^{-1} [e_1 \dots \hat{e}_i \dots e_{k+1}] \left((1 - \sum_{j=1}^{k-1} x_j) e_1 + \sum_{j=2}^{k} x_{j-1} e_j \right)$$
$$= \rho_k^{-1} \left((1 - \sum_{j=1}^{k-1} x_j) e_1 + \sum_{j=2}^{i-1} x_{j-1} e_j + \sum_{j=i}^{k} x_{j-1} e_{j+1} \right)$$

Since $\quad f_{k-1}^{-1} = [0, e_1, \ldots, e_{k-1}] : \Delta_{k-1} \to \Delta_k^{\circ}$

(check : $[0, e_1 \ldots e_{k-1}] \circ f_k \ (x_1 \ldots x_k)$

$$= [0\ e_1 \ldots e_{k-1}] \left((1 - \sum_i x_i)\, e_1 + \sum_{j=2}^{k+1} x_{j-1} e_j \right)$$

$$= \sum_{j=2}^{k+1} x_{j-1}\, e_{j-1} = \sum_{j=1}^{k} x_j\, e_j = (x_1 \ldots x_k) \quad)$$

we have

$$F_i(x_1, \ldots x_{k-1}) = [0, e_1, \ldots e_{i-1}, e_{i+1} \ldots e_{k}] \left((1 - \sum x_j) e_1 + \sum_{j=2}^{i-1} x_{j-1} e_j + \sum_{j=i}^{k} x_{j-1} e_j \right)$$

$$= \sum_{j=2}^{i-1} x_{j-1}\, e_{j-1} + \sum_{j=i}^{k} x_{j-1}\, e_j$$

$$= (x_1, \ldots, x_{i-2}, \underset{\substack{\uparrow \\ (i-1)^{th}}}{0}, x_{j-1}, \ldots, x_{k-1})$$

Thus $\quad F_i^*(dy_1 \wedge \ldots \wedge \widehat{dy_\ell} \wedge \ldots \wedge dy_k)$

$$= \begin{cases} dx_1 \wedge \ldots \wedge dx_{k-1} & \text{if } \ell = i-1 \\ 0, & \text{otherwise.} \end{cases}$$

for $i > 1$.

If $i = 1$, $\quad F_1(x_1 \ldots x_{k-1}) = f_k^{-1}\, [e_2 \ldots e_{k+1}]\, f_{k-1}\, (x_1 \ldots x_{k-1})$

$$= f_k^{-1}\, [e_2 \ldots e_{k+1}] \left((1 - \sum_{j=1}^{k-1} x_j) e_1 + \sum_{j=2}^{k} x_{j-1} e_j \right)$$

$$= f_k^{-1} \left((1 - \sum_{j=1}^{k-1} x_j) e_2 + \sum_{j=2}^{k} x_{j-1} e_{j+1} \right)$$

$$= [0, e_1 \ldots e_k] \left((1 - \sum_{j=1}^{k-1} x_j) e_2 + \sum_{j=3}^{k+1} x_{j-2} e_j \right)$$

$$= (1 - \sum_{j=1}^{k-1} x_j) e_1 + \sum_{j=3}^{k+1} x_{j-2}\, e_{j-1}$$

$$= (1 - \sum_{j=1}^{k-1} x_j) e_1 + \sum_{j=2}^{k} x_{j-1}\, e_j$$

$$= \left((1 - \sum_{j=1}^{k-1} x_j), x_1, x_2 \ldots x_{k-1} \right)$$

Thus $\quad F_1^*(dy_1 \wedge \ldots \wedge \widehat{dy_\ell} \wedge \ldots \wedge dy_k)$

$$= \begin{cases} dx_1 \wedge \ldots \wedge dx_{k-1}, & \ell = 1 \\ -\sum_{j=1}^{k-1} dx_j \wedge dx_1 \wedge \ldots \wedge \widehat{dx_{\ell-1}} \wedge \ldots \wedge dx_{k-1} & \ell > 1 \end{cases}$$

Thus $F_1^*(dy_1 \wedge \cdots \wedge \widehat{dy_\ell} \wedge \cdots \wedge dy_{k-1})$

$$= \begin{cases} dx_1 \wedge \cdots \wedge dx_{k-1}, & \ell = 1 \\ (-1)^{\ell-1} dx_1 \wedge \cdots \wedge dx_{k-1}, & \ell \geq 2. \end{cases}$$

$$= (-1)^{\ell-1} dx_1 \wedge \cdots \wedge dx_{k-1}$$

Step 3: Let $\rho_k^* \sigma^* \omega = \sum_{\ell=1}^{k} f_\ell \, dy_1 \wedge \cdots \wedge \widehat{dy_\ell} \wedge \cdots \wedge dy_k$

Then

$$\int_{\partial\sigma} \omega = \sum_{i=1}^{k+1} (-1)^{i-1} \int_{\Delta_{k-1}^o} F_i^* \rho_k^* \sigma^* \omega$$

$$= \sum_{i=1}^{k+1} (-1)^{i-1} \int_{\Delta_{k-1}^o} F_i^* \left(\sum_{\ell=1}^{k} f_\ell \, dy_1 \wedge \cdots \wedge \widehat{dy_\ell} \wedge \cdots \wedge dy_k \right)$$

$$= \int_{\Delta_{k-1}^o} F_1^* \left(\sum_{\ell=1}^{k} f_\ell \, dy_1 \wedge \cdots \wedge \widehat{dy_\ell} \wedge \cdots \wedge dy_k \right)$$

$$+ \sum_{i=2}^{k+1} (-1)^{i-1} \int_{\Delta_{k-1}^o} F_i^* \left(\sum_{\ell=1}^{k} f_\ell \, dy_1 \wedge \cdots \wedge \widehat{dy_\ell} \wedge \cdots \wedge dy_k \right)$$

$\ell = i-1$ surviving

$\iff i = \ell+1$ surviving

$$= \sum_{\ell=1}^{k} (-1)^{\ell-1} \int_{\Delta_{k-1}^o} f_\ell \circ F_1 \, dx_1 \wedge \cdots \wedge dx_{k-1}$$

$$+ \sum_{\ell=1}^{k} (-1)^{\ell} \int_{\Delta_{k-1}^o} f_\ell \circ F_{\ell+1} \, dx_1 \wedge \cdots \wedge dx_{k-1}$$

$$= \sum_{\ell=1}^{k} (-1)^{\ell-1} \int_{\Delta_{k-1}^o} (f_\ell \circ F_1 - f_\ell \circ F_{\ell+1}) \, dx_1 \wedge \cdots \wedge dx_{k-1}$$

Step 4:

$$(-1)^{\ell-1}\int_{\Delta^0_{k-1}} (f_\ell\circ F_1 - f_\ell\circ F_{\ell+1})\, dx_1\wedge\dots\wedge dx_{k-1}$$

$$= (-1)^{\ell-1}\int_{\Delta^0_{k-1}} \Big(f_\ell(\overbrace{1-\sum_{j=1}^{k-1}x_j,\ x_1,\dots,x_{k-1}}^{\in}) - f_\ell(\overbrace{x_1,\dots,x_{\ell-1},\underset{\substack{\uparrow\\ \ell\text{'s}}}{0},x_\ell,\dots,x_{k-1}}^{\in})\Big)\, d\alpha^{k-1}$$

Change of variables

$$= (-1)^{\ell-1}\int_{\Delta^0_{k-1}} \Big(f_\ell(x_1,\dots,x_{\ell-1},1-\sum_{j=1}^{k-1}x_j,x_\ell,\dots,x_{k-1}) - f_\ell(x_1,\dots,x_{\ell-1},0,x_\ell,\dots,x_{k-1})\Big)\, d\alpha^{k-1}$$

$$= (-1)^{\ell-1}\int_{\Delta^0_{k-1}} \Big(\int_0^{1-\sum_{j=1}^{k-1}x_j} \frac{\partial f_\ell}{\partial x_\ell}(x_1,\dots,x_{\ell-1},t,x_\ell,\dots,x_{k-1})\,dt\Big)\, d\alpha^{k-1}$$

$$= (-1)^{\ell-1}\int_{\Delta^0_k} \frac{\partial f_\ell}{\partial x_\ell}\, dx_1\wedge\dots\wedge dx_{k-1} = \int_{\Delta^0_k} d(f_\ell\, dx_1\wedge\dots\wedge dx_{k-1})$$

Hence
$$\int_{\partial\sigma}\omega = \sum_{\ell=1}^{k}(-1)^{\ell-1}\int_{\Delta^0_{k-1}}(f_\ell\circ F_1 - f_\ell\circ F_{\ell+1})\, dx_1\wedge\dots\wedge dx_{k-1}$$

$$= \sum_{\ell=1}^{k}\int_{\Delta^0_k} d(f_\ell\, dx_1\wedge\dots\wedge dx_{k-1}) = \int_{\Delta^0_k} d\, \rho_k^*\sigma^*\omega$$

$$= \int_{\Delta^0_k}\rho_k^*\sigma^*\, d\omega = \int_\sigma d\omega$$

□

Corollary: Suppose ω closed k-form, σ a k-cycle, τ a $(k-1)$-form and γ $(k+1)$-chain. Then

$$\int_{\sigma+\partial\gamma}\omega + d\tau = \int_\sigma \omega$$

Pf:
$$\int_{\sigma+\partial\gamma}\omega + d\tau = \int_\sigma\omega + \int_{\partial\gamma}\omega + \int_\sigma d\tau + \int_{\partial\gamma} d\tau$$

$$= \int_\sigma\omega + \int_\gamma d\omega + \int_{\partial\sigma}\tau + \int_\gamma d^2\tau = \int_\sigma\omega$$

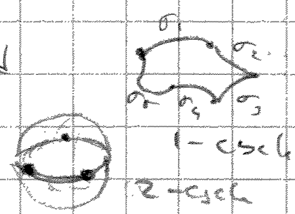
□

Let X be a top. space

Def: $Z_k(X) = \{\sigma \in C_k(X) : \partial\sigma = 0\} = \ker(\partial : C_k \to C_{k-1})$

$B_k(X) = \{\partial\sigma \in C_k(X) : \sigma \in C_{k+1}(X)\} = \operatorname{Im}(\partial : C_{k+1} \to C_k)$

Elements in $Z_k(X)$ are called *k-cycles* and elements in $B_k(X)$ *k-boundaries*.



If $X \subset \mathbb{R}^n$ open (or a manifold) we define

Def: $Z_k^\infty(X) = \{\sigma \in Z_k(X) : \sigma \circ f_k \text{ is } C^\infty\text{-smooth}\}$

and $B_k^\infty(X) = \{\sigma \in B_k(X) : \sigma \circ f_k \text{ is } C^\infty\text{-smooth}\}$

Since $\partial^2 = 0$, $B_k(X) \subset Z_k(X)$ for all k,
we define the homology of the complex $(C_k(X))_{k \geq 0}$ by

$$H_k(X) = Z_k(X) / B_k(X).$$

If $U \subset \mathbb{R}^n$ open (or a manifold) we define

$$H_k^\infty(U) = Z_k(U) / B_k(U)$$

In this case we also define the k^{th} de Rham cohomology of

$$H^k_{dR}(U) = \frac{\ker\{d : \Omega^k(U) \to \Omega^{k+1}(U)\}}{\operatorname{Im}\{d : \Omega^k(U) \to \Omega^k(U)\}} \left(= \frac{\{\text{closed } k\text{-forms}\}}{\{\text{exact } k\text{-forms}\}} \right)$$

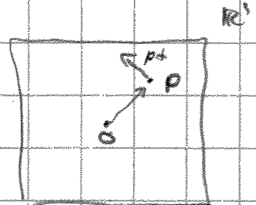
Then $H^k_{dR}(U)$ is a vector space and we have a reformulation of the previous corollary.

Cor: The pairing $Z_k^\infty(X) \times \Omega^k_{closed}(X) \longrightarrow \mathbb{R}$, $(\sigma, \omega) \longmapsto \int_\sigma \omega$
decends to a pairing

$$H_k^\infty(X) \times H^k_{dR}(X) \longrightarrow \mathbb{R}$$
$$([\sigma], [\omega]) \longmapsto \int_\sigma \omega.$$

<u>Example:</u> $H^1(\mathbb{R}^2 \setminus \{0\}) \neq 0$ (= Exercise 1.1)

Take $\omega_p(v) = \langle v, \frac{p^\perp}{|p|^2} \rangle$



where $p^\perp = (p_1, p_2)^\perp = (-p_2, p_1)$

i.e. $\langle p^\perp, p \rangle = -p_2 \cdot p_1 + p_1 \cdot p_2 = 0$.

Then $\omega_x(v) = \frac{1}{|x|^2} \langle v, x^\perp \rangle = \frac{1}{|x|^2}\left(v_1 \cdot (-x_2) + v_2 x_1 \right)$

$$= \frac{1}{|x|^2}\left(x_1\, dx_2(v) - x_2\, dx_1(v) \right)$$

$$= \left(\frac{x_1}{|x|^2} dx_2 - \frac{x_2}{|x|^2} dx_1 \right)(v).$$

Moreover, $d\omega = d\left(\frac{x_1}{|x|} dx_2 - \frac{x_2}{|x|} dx_1 \right)$

$$= \frac{1}{|x|^2} dx_1 \wedge dx_2 - 2\frac{x_1^2}{|x|^4} dx_1 \wedge dx_2$$

$$- \frac{1}{|x|^2} dx_2 \wedge dx_1 + 2\frac{x_2^2}{|x|^4} dx_2 \wedge dx_1$$

$$= \frac{2}{|x|^2} dx_1 \wedge dx_2 - 2\frac{x_1^2 + x_2^2}{|x|^4} dx_1 \wedge dx_2$$

$$= \frac{2}{|x|^2} dx_1 \wedge dx_2 - \frac{2}{|x|^2} dx_1 \wedge dx_2 = 0$$

$$d\left(\frac{1}{|x|^2}\right) = d\left((x_1^2 + x_2^2)^{-1}\right) = (-1)(x_1^2 + x_2^2)^{-2} \cdot (2x_1\, dx_1 + 2x_2\, dx_2) = -2\frac{x_1}{|x|^4} dx_1 - 2\frac{x_2}{|x|^4} dx_2$$

Thus ω is closed.

Let $\gamma : [0,1] \to \mathbb{R}^2$ $\gamma(t) = (\cos 2\pi t, \sin 2\pi t)$

Then $\sigma : \Delta_1 \to \mathbb{R}^2$ $\sigma(1-t, t) = \gamma(t)$ is a 1-chain

and $\partial\sigma = \sigma(e_2) - \sigma(e_1) = \gamma(1) - \gamma(0) = (1,0) - (1,0) = 0$.

Thus σ is a cycle.

Since $\int_\sigma \omega \underset{\text{check!}}{=} \int_\gamma \omega = \int_{[0,1]} \gamma^* \omega \underset{\text{check!}}{=} \int_{[0,1]} \omega_{\gamma(t)}(\gamma'(t))\, dt$

$$= \int_{[0,1]} 2\pi \langle (-\sin 2\pi t, \cos 2\pi t), (-\sin 2\pi t, \cos 2\pi t) \rangle\, dt$$

$$= \int_{[0,1]} 2\pi\, dt = 2\pi \neq 0 \qquad \square$$