Multilinear algebra

Let V be a finite dimensional vector space.

Def: A map $\varphi : \overbrace{V\times\dots\times V}^{k \text{ times}} \longrightarrow \mathbb{R}$ is <u>k-linear</u> if
φ is linear in each argument, i.e.,
for all $v_1, \dots, v_k \in V$ and all $1 \le l \le k$
the map $V \longrightarrow \mathbb{R}$, $v \longmapsto \varphi(v_1, \dots, v_{l-1}, v, v_{l+1}, \dots, v_k)$,
is linear.

Rmk: Convention is that for $k=0$, $V^k = \mathbb{R}$ and
$\varphi : V^0 \longrightarrow \mathbb{R}$ 0-linear $\Leftrightarrow$ φ linear.

Def: A k-linear map $\varphi : V^k \longrightarrow \mathbb{R}$ is <u>alternating</u>
if $\varphi(v_1, \dots, v_k) = 0$ whenever $v_i = v_j$ for $i \ne j$.

Lemma: Let $k \geq 2$ and $\sigma : \{1, \dots, k\} \to \{1, \dots, k\}$ be a transposition,
i.e. σ is a bijection and $\exists\ i \ne j$ s.t.

$$\sigma(l) = \begin{cases} j, & \text{if } l = i \\ i, & \text{if } l = j \\ l, & \text{otherwise.} \end{cases}$$

Let $\varphi : V^k \longrightarrow \mathbb{R}$ be alternating k-linear map.
Then
$$\varphi(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = -\varphi(v_1, \dots, v_k)$$
for all $v_1, \dots, v_k \in V$.

Cor: $\varphi(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sign}(\sigma)\, \varphi(v_1, \dots, v_k)$
for all permutations $\sigma : \{1 \dots k\} \to \{1 \dots k\}$.

Pf of Lemma: Set $w_l = v_{\sigma(l)} + v_l$. Then $w_i = w_j$ and $w_l = 2v_l$ for $l \ne i, j$.

Thus
$$\begin{aligned} 0 = \varphi(w_1, \dots, w_k) &= \varphi(w_1, \dots, v_{\sigma(i)} + v_i, \dots, w_k) \\ &= \varphi(w_1 \dots v_{\sigma(i)}, \dots, v_{\sigma(j)} + v_j, \dots w_k) \\ &\quad + \varphi(w_1, \dots, v_i, \dots, v_{\sigma(j)} + v_j, \dots w_k) \\ &= \varphi(w_1, \dots, v_j, \dots, v_i, \dots w_k) + 0 + 0 + \varphi(w_1, \dots, v_i, \dots, v_j, \dots w_k) \\ &= 2^{k-2} \big(\varphi(v_1, \dots, v_j, \dots, v_i, \dots, v_k) + \varphi(v_1, \dots, v_i, \dots v_j, \dots v_k) \big) \;\square \end{aligned}$$

Pf of Cor: Every permutation σ is a composition of transpositions and $\operatorname{sign}(\sigma) = (-1)^{\#\text{transpositions}}$ is well defined. $\triangledown$

Example 1: $\det : \overbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}^{n \text{ times}} \longrightarrow \mathbb{R}$

$$\det(v_1, \dots, v_n) = \det \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

is alternating and n-linear.

Exercise: Take $\det \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \end{bmatrix}}_{=A} = \sum_{k=1}^{n} (-1)^k a_{1k} \det \begin{bmatrix} a_{11} & \cdots & a_{1k} & \cdots & a_{1n} \\ \vdots & & & & \vdots \\ a_{n1} & \cdots & a_{nk} & \cdots & a_{nn} \end{bmatrix}$ remove

as the recursive definition of the determinant.
Show that
$$\det A = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$
where S_n is the group of permutations of $\{1, \dots, n\}$.

Notation: We denote by $\mathrm{Alt}^k(V)$ the set of all alternating k-linear maps on V.

Obs: $\mathrm{Alt}^0(V) = L(\mathbb{R}, \mathbb{R})$ and $\mathrm{Alt}^1(V) = L(V, \mathbb{R}) = V^*$.

Lemma: Let W be a fin. dim. vectorspace and $f: W \to V$ linear map.
Then, for every $\varphi \in \mathrm{Alt}^k(V)$, the map
$$f^*\varphi : W \times \dots \times W \longrightarrow \mathbb{R} \text{ defined by}$$
$W \xrightarrow{f} V$
$$(f^*\varphi)(w_1, \dots, w_k) = \varphi(f(w_1), \dots, f(w_k))$$
$\mathrm{Alt}^k(W) \xleftarrow{f^*} \mathrm{Alt}^k(V)$

is alternating and k-linear, i.e. $f^*\varphi \in \mathrm{Alt}^k(W)$.
In particular, $\varphi \longmapsto f^*\varphi$ defines a map $f^* : \mathrm{Alt}^k(V) \longrightarrow \mathrm{Alt}^k(W)$

Pf: Suppose $w_i = w_j$ for $i \neq j$. Then $f(w_i) = f(w_j)$.
Thus
$$(f^*\varphi)(w_1, \dots, w_k) = \varphi(f(w_1), \dots, f(w_k)) = 0,$$
i.e. $f^*\varphi$ is "alternating" (but not yet k-linear)

$f^*\varphi$ is k-linear: Let $w_1, \dots, w_k \in W$, $w, w' \in W$, $a \in \mathbb{R}$.
Then $f^*\varphi(w_1, \dots, \overset{i\text{th}}{aw + w'}, \dots, w_k) = \varphi(f(w_1), \dots, f(aw + w'), \dots, f(w_k))$

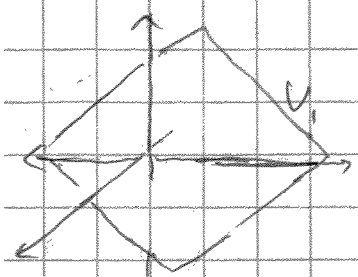
f linear $\longrightarrow = \varphi(f(w_1), \dots, af(w) + f(w'), \dots, f(w_k))$
$= a\varphi(f(w_1), \dots, f(w), \dots, f(w_k)) + \varphi(f(w_1), \dots, f(w'), \dots, f(w_k))$
$= a\,(f^*\varphi)(w_1, \dots, w, \dots, w_k) + f^*\varphi(w_1, \dots, w', \dots, w_k)$. $\square$

Example 2: Let $V_1, \dots, V_m$ be k-dimensional subspaces of V

$V = \mathbb{R}^3$

Let $\pi_\ell : V \longrightarrow V_\ell$ be projection and

let $f_\ell : V_\ell \longrightarrow \mathbb{R}^k$ be isomorphism.



Then $f_\ell^* \det$ is an alternating k-linear map on V_ℓ for every ℓ and $\pi_\ell^* f_\ell^* \det$ is an alternating k-linear map on V for every ℓ

Lemma: Let $f : Y \longrightarrow W$ and $g : W \longrightarrow V$ be linear maps and $\varphi : V \times \dots \times V \to \mathbb{R}$ alternating k-linear form. Then

$$f^* g^* \varphi = (g \circ f)^* \varphi$$

Pf:
$$\begin{aligned} f^* g^* \varphi(y_1, \dots, y_k) &= g^* \varphi(f(y_1), \dots, f(y_k)) \\ &= \varphi(g(f(y_1)), \dots, g(f(y_k))) \\ &= \varphi((g\circ f)(y_1), \dots, (g\circ f)(y_k)) \\ &= (g \circ f)^* \varphi(y_1, \dots, y_k) \end{aligned}$$
$\square$

Lemma: Let $\varphi, \psi : V \times \dots \times V \to \mathbb{R}$ be alternating k-linear maps and $a \in \mathbb{R}$. Then

$$(a\varphi + \psi) : V \times \dots \times V \to \mathbb{R}$$
$$(a\varphi + \psi)(v_1, \dots, v_k) = a\,\varphi(v_1, \dots, v_k) + \psi(v_1, \dots, v_k)$$

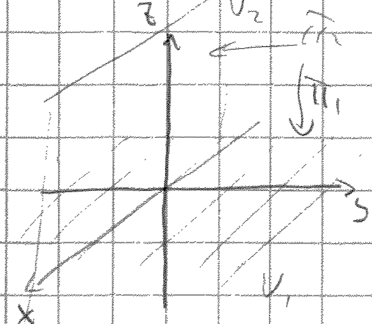
is alternating and k-linear.

Pf: Exercise $\square$

Cor: Alternating k-linear maps on V form a vector space.

Example 2 (cont) $\quad \psi = \sum_\ell (f_\ell \circ \pi_\ell)^* \det$ is alternating and k-linear

Example): $V = \mathbb{R}^3 \quad V_1 = \{0\} \times \mathbb{R}^2, \quad V_2 = \mathbb{R} \times \{0\} \times \mathbb{R}, \quad V_3 = \mathbb{R}^2 \times \{0\}$



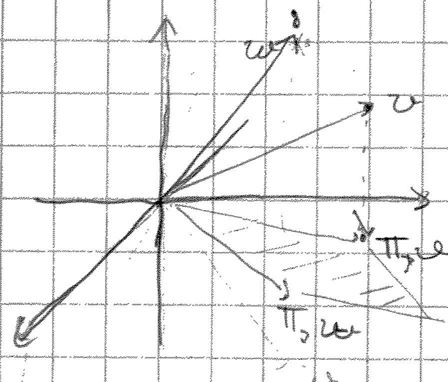
$$\pi_1(x,y,z) = (0,y,z)$$
$$\pi_2(x,y,z) = (x,0,z)$$
$$\pi_3(x,y,z) = (x,y,0)$$

$$f_1(0,y,z) = (y,z)$$
$$f_2(x,0,z) = (x,z)$$
$$f_3(x,y,0) = (x,y)$$

$$\pi_1^* f_1^* \det\left(\underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}}_{\in V_1}\right) = f_1^* \det\left(\begin{bmatrix} 0 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} 0 \\ w_2 \\ w_3 \end{bmatrix}\right)$$

$$= \det \begin{bmatrix} v_2 & w_2 \\ v_3 & w_3 \end{bmatrix}$$

$$\pi_2^* f_2^* \det\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}\right) = f_2^* \det\left(\begin{bmatrix} v_1 \\ 0 \\ v_3 \end{bmatrix} \begin{bmatrix} w_1 \\ 0 \\ w_3 \end{bmatrix}\right) = \det \begin{bmatrix} v_1 & w_1 \\ v_3 & w_3 \end{bmatrix}$$



$\square$

In general: Let $0 \le k \le n$ and $I = (i_1, \dots, i_k)$ where $1 \le i_1 < i_2 < \dots < i_k \le n$.

Denote $\mathbb{R}^n_I = \{ (v_1, v_2, \dots, v_n) \in \mathbb{R}^n : v_\ell = 0 \text{ if } \ell \neq i_j \text{ for all } j \}$

Example: $\mathbb{R}^3_{\{1,2\}} = \mathbb{R} \times \mathbb{R} \times \{0\} = \mathbb{R}^2 \times \{0\}$
$\mathbb{R}^3_{\{2\}} = \{0\} \times \mathbb{R} \times \{0\}$

Denote $f_I : \mathbb{R}^n \longrightarrow \mathbb{R}^k$
$(v_1, \dots, v_n) \longmapsto (v_{i_1}, \dots, v_{i_k})$

Thm: (Geometric Structure Theorem for Alt^k) Let $\varphi : \overbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}^{k \text{ times}} \longrightarrow \mathbb{R}$ be alternating and k-linear. Then $\exists$ constants $a_I \in \mathbb{R}$ for $I = (i_1 \dots i_k)$, $1 \le i_1 < \dots < i_k \le n$ so that

$$\varphi = \sum a_I \cdot f_I^* \det$$

where $\det$ is the standard determinant in $\mathbb{R}^k$.

Proof: (Exercise) $\square$

Alternating 1-linear maps

$\varphi : V \longrightarrow \mathbb{R}$ alternating 1-linear $\iff$ $\varphi : V \longrightarrow \mathbb{R}$ linear

Denote $\mathcal{L}(V, \mathbb{R})$ linear maps from V to $\mathbb{R}$

Thm: Let $\{e_1, \dots, e_n\}$ be a basis of V.
Then maps $\varepsilon_k : V \longrightarrow \mathbb{R}$, $\sum_l a_l v_l \longmapsto a_k$

form a basis of $\mathcal{L}(V, \mathbb{R})$, i.e., for every $f \in \mathcal{L}(V, \mathbb{R})$ exists constants $a_1, \dots, a_n \in \mathbb{R}$ s.t.

$$f = \sum_k a_k \varepsilon_k \quad \text{and} \quad a_k = f(e_k).$$

Pf: Linear algebra. $\square$

Rmks: $\mathcal{L}(V, \mathbb{R})$ is called the dual space of V and usually denoted V^*.
Basis $\{\varepsilon_1, \dots, \varepsilon_n\}$ of V^* is called the dual basis.

Alternating 2-linear maps

Let $\{e_1, \dots, e_n\}$ be a basis of V and $\{\varepsilon_1, \dots, \varepsilon_n\}$ the dual basis.

Let $f, g \in \mathcal{L}(V, \mathbb{R})$. Define $f \wedge g : V \times V \longrightarrow \mathbb{R}$ by

$$f \wedge g(v_1, v_2) = f(v_1)\, g(v_2) - f(v_2)\, g(v_1)$$

Lemma: $f \wedge g$ is alternating and 2-linear.

Pf:
- $f \wedge g(v, v) = 0$ clear
- $f \wedge g(a v + w, v_2) = (a f(v) + f(w))\, g(v_2) - f(v_2)(a g(v) + g(w))$
$= a f(v) g(v_2) + f(w) g(v_2) - a f(v_2) g(v) - f(v_2) g(w)$

<u>Lemma:</u> $\varphi : V \times V \to \mathbb{R}$ alternating 2-linear map.
Then
$$\varphi = \sum_{1 \le i < j \le n} a_{ij}\, \varepsilon_i \wedge \varepsilon_j \quad \text{and} \quad a_{ij} = \varphi(e_i, e_j)$$

<u>Pf:</u> Let $v = \sum_i c_i e_i$, $w = \sum_i c_i' e_i$

Then
$$\varphi(v, w) = \varphi\Big(\sum_i c_i e_i, \sum_j c_j' e_j\Big)$$
$$= \sum_i c_i\, \varphi\Big(e_i, \sum_j c_j' e_j\Big)$$
$$= \sum_{ij} c_i c_j'\, \varphi(e_i, e_j)$$

Since $\varphi(e_i, e_j) = -\varphi(e_j, e_i)$,
$$\varphi(v, w) = \sum_{ij} c_i c_j'\, \varphi(e_i, e_j)$$
$$= \sum_{i<j} (c_i c_j' - c_j c_i')\, \varphi(e_i, e_j)$$

Denote $a_{ij} = \varphi(e_i, e_j)$.

On the other hand,
$$\varepsilon_i \wedge \varepsilon_j (v, w) = \varepsilon_i(v)\, \varepsilon_j(w) - \varepsilon_i(w)\, \varepsilon_j(v)$$
$$= c_i c_j' - c_i' c_j$$

Thus
$$\varphi(v, w) = \sum_{i<j} a_{ij}\, \varepsilon_i \wedge \varepsilon_j (v, w)$$
$$= \Big(\sum_{i<j} a_{ij}\, \varepsilon_i \wedge \varepsilon_j\Big)(v, w).$$

$\square$

Alternating k-linear maps

Let $\{e_1 \dots e_n\}$ be a basis of V and $\{\varepsilon_1 \dots \varepsilon_n\}$ dual basis.

Let $f_1 \dots f_k : V \longrightarrow \mathbb{R}$ linear. Define
$f_1 \wedge \dots \wedge f_k : V \times \dots \times V \longrightarrow \mathbb{R}$ by

$$f_1 \wedge \dots \wedge f_k (v_1, \dots, v_k) = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\, f_1(v_{\sigma(1)}) \cdots f_k(v_{\sigma(k)})$$

where S_k is the permutation group of $\{1 \dots k\}$

Lemma: $f_1 \wedge \dots \wedge f_k$ is alternating and k-linear

Recall: Let $A = \begin{bmatrix} a_{11} & \dots & a_{1k} \\ a_{k1} & \dots & a_{kk} \end{bmatrix}$. Then

$$\det A = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} \cdots a_{k\sigma(k)}.$$

Pf of lemma: Define $F : V \longrightarrow \mathbb{R}^k$, $F = (f_1 \dots f_k)$. Then

$$f_1 \wedge \dots \wedge f_k (v_1, \dots, v_k) = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\, f_1(v_{\sigma(1)}) \cdots f_k(v_{\sigma(k)})$$

$$= \det \begin{bmatrix} f_1(v_1) & f_1(v_2) & \dots & f_1(v_k) \\ \vdots & \vdots & & \vdots \\ f_k(v_1) & f_k(v_2) & & f_k(v_k) \end{bmatrix}$$

$$= \det \begin{bmatrix} F(v_1) & \dots & F(v_k) \end{bmatrix}. \qquad \square$$

Observation: $f_1 \wedge \dots \wedge f_k (v_1, \dots, v_k) = \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\, f_{\sigma(1)}(v_1) \cdots f_{\sigma(k)}(v_k)$

Representation and basis

Let $S(n,k) = \{ (i_1, \dots, i_k) : 1 \le i_1 < \dots < i_k \le n \}$.

Given $I \in S(n,k)$, $I = (i_1, \dots, i_k)$,
We set
$$\varepsilon_I = \varepsilon_{i_1} \wedge \dots \wedge \varepsilon_{i_k}.$$

Thm: Let $\varphi : V \times \cdots \times V \to \mathbb{R}$ be alternating and k-linear. Then

$$\varphi = \sum_{I \in S(n,k)} \varphi(e_{i_1} \cdots e_{i_k})\, \varepsilon_I$$

We denote by $Alt^k(V)$ the space of alternating k-linear maps on V.

Cor: $\{\varepsilon_I : I \in S(n,k)\}$ is a basis of $Alt^k(V)$.

Pf of Thm: Let $v_1 \ldots v_k \in V$ and $v_i = \sum_j v_{ij} e_j$. Then

$$\varphi(v_1, \ldots, v_k) = \varphi\Big(\sum_{i_1} v_{1 i_1} e_{i_1}, \ldots, \sum_{i_k} v_{k i_k} e_{i_k}\Big)$$

$$= \sum_{i_1 \cdots i_k} v_{1 i_1} \cdots v_{k i_k}\, \varphi(e_{i_1}, \ldots, e_{i_k})$$

Note that $v_{1 i_1} \cdots v_{k i_k} = \varepsilon_{i_1}(v_1) \cdots \varepsilon_{i_k}(v_k)$

Thus $\varphi(v_1 \ldots v_k) = \sum_{i_1 \cdots i_k} \varphi(e_{i_1}, \ldots, e_{i_k})\, \varepsilon_{i_1}(v_1) \cdots \varepsilon_{i_k}(v_k)$

$$= \sum_{J = (j_1 \ldots j_k) \in S(n,k)} \sum_{\sigma \in S_k} \varphi(e_{j_{\sigma(1)}} \cdots e_{j_{\sigma(k)}})\, \varepsilon_{j_{\sigma(1)}}(v_1) \cdots \varepsilon_{j_{\sigma(k)}}(v_k)$$

$$= \sum_{J = (j_1 \cdots j_k)} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\, \varphi(e_{j_1}, \ldots, e_{j_k})\, \varepsilon_{j_{\sigma(1)}}(v_1) \cdots \varepsilon_{j_{\sigma(k)}}(v_k)$$

$$= \sum_{J = (j_1 \cdots j_k)} \varphi(e_{j_1} \cdots e_{j_k}) \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\, \varepsilon_{j_{\sigma(1)}}(v_1) \cdots \varepsilon_{j_{\sigma(k)}}(v_k)$$

$$= \sum_J \varphi(e_{j_1}, \ldots, e_{j_k}) \sum_{\sigma \in S_k} \operatorname{sign}(\sigma^{-1})\, \varepsilon_1(v_{\sigma^{-1}(1)}) \cdots \varepsilon_k(v_{\sigma^{-1}(k)})$$

$\square$

Pf of Cor: $\{\varepsilon_I\}$ clearly generates $Alt^k(V)$.

Linear independence exercise

Observation: Let $(j_1 \dots j_k) \in \{1 \dots n\}^k$. Then

$$\varepsilon_{j_1} \wedge \dots \wedge \varepsilon_{j_k} = \begin{cases} 0 & \text{if } \#\{j_1 \dots j_k\} < k \\ \operatorname{sign}(\sigma)\, \varepsilon_I, & \text{if } I = (j_{\sigma(1)} \dots j_{\sigma(k)}) \end{cases}$$

where $\sigma : \{j_1 \dots j_k\} \to \{j_1 \dots j_k\}$ so that $\sigma(j_\ell) = i_\ell$. □

Suppose $\dim V = n$. Then $\mathrm{Alt}^k(V) = \{0\}$ for $k > n$.
Moreover $\dim \mathrm{Alt}^n(V) = 1$ and $f^*: \mathrm{Alt}^n(V) \to \mathrm{Alt}^n(V)$

$f^*\varphi = (\det f)\varphi$

Pf: $f^*(\varepsilon_1 \wedge \dots \wedge \varepsilon_n)(e_1, \dots, e_n)$

$= (\varepsilon_1 \wedge \dots \wedge \varepsilon_n)(f(e_1), \dots f(e_n))$

$= \sum_\sigma \operatorname{sign}(\sigma)\, \varepsilon_1(f(e_{\sigma(1)})) \cdots \varepsilon_n(f(e_{\sigma(n)}))$

$= \sum_\sigma \operatorname{sign}(\sigma)\, f_{\sigma(1)1} \cdots f_{\sigma(n)n} = \det f$

General products

Let $\varphi \in \mathrm{Alt}^k(V)$ and $\psi \in \mathrm{Alt}^\ell(V)$.
Define

$$\varphi \wedge \psi : \overbrace{V \times \dots \times V}^{k+\ell} \longrightarrow \mathbb{R}$$

$$\varphi \wedge \psi(v_1 \dots v_{k+\ell}) = \sum_{\sigma \in S_{k+\ell,k}} \operatorname{sign}(\sigma)\, \varphi(v_{\sigma(1)} \dots v_{\sigma(k)})\, \psi(v_{\sigma(k+1)} \dots v_{\sigma(k+\ell)})$$

where

$$S_{k+\ell} = \{\sigma \in S_n : \sigma(1) < \dots < \sigma(k) \text{ and } \sigma(k+1) < \dots < \sigma(n)\}$$

Lemma: $\varphi \wedge \psi \in \mathrm{Alt}^{k+\ell}(V)$

Pf: $(k+\ell)$-linearity ok.
Suppose $v_i = v_j$ for $i < j$. Let τ be the transposition $i \leftrightarrow j$.

Let $\sigma \in S_{k+\ell,k}$. If $\sigma^{-1}(i) \le k$ and $\sigma^{-1}(j) \le k$
or $\sigma^{-1}(i) > k$ and $\sigma^{-1}(j) > k$
then $\varphi(v_{\sigma(1)} \dots v_{\sigma(k)})\, \psi(v_{\sigma(k+1)} \dots v_{\sigma(k+\ell)}) = 0$.

If $\sigma^{-1}(i) \le k < \sigma^{-1}(j)$, then $(\tau\circ\sigma)^{-1}(i) > k \ge (\tau\circ\sigma)^{-1}(j)$
and $\operatorname{sign}(\tau\circ\sigma)\, \varphi(v_{(\tau\circ\sigma)(1)} \dots v_{(\tau\circ\sigma)(k)})\, \psi(v_{(\tau\circ\sigma)(k+1)} \dots v_{(\tau\circ\sigma)(k+\ell)})$

$$= -\operatorname{sign}(\sigma)\, \varphi(v_{\sigma(1)} \dots v_{\sigma(k)})\, \psi(v_{\sigma(k+1)} \dots v_{\sigma(k+\ell)})$$

Thus

$$\sum_{\sigma \in S_{k+\ell,k}} = \sum_{\substack{\sigma \in S_{k+\ell,k} \\ \sigma^{-1}(i) \le k \\ \sigma^{-1}(j) > k}} + \sum_{\substack{\sigma \in S_{k+\ell,k} \\ \sigma^{-1}(i) > k \\ \sigma^{-1}(j) \le k}} = 0.$$

□

Observation: Let $f: W \to V$ be a linear map and $\omega \in Alt^k(V)$ and $\tau \in Alt^l(V)$. Then
$f^*(\omega) \wedge f^*(\tau) = f^*(\omega \wedge \tau) \in Alt^{k+l}(W)$.
(Use the definitions)
In particular, $f^*(\omega_1 \wedge \dots \wedge \omega_k) = f^*\omega_1 \wedge \dots \wedge f^*\omega_k$
for $\omega_1, \dots, \omega_k \in Alt^1(V)$.

Exercise: Let $L: \mathbb{R}^n \to \mathbb{R}^n$ be a linear map, so that
$L^*: Alt^2(\mathbb{R}^n) \to Alt^2(\mathbb{R}^n)$ is identity i.e. $L^*\varphi = \varphi$.
What can we say about L when
(a) $n=2$
(b) $n=3$
(c) $n=4$.

Excursion

Free vectorspace over a set.

Let S be a set and denote by $\mathbb{R}^S$ the vectorspace of all functions $S \to \mathbb{R}$.

We call the space

$$F(S) = \operatorname{span}\{\chi_{\{s\}} : s \in S\} \subset \mathbb{R}^S,$$

where $\chi_{\{s\}}$ is the characteristic function $\chi_{\{s\}}(x) = \begin{cases} 1, & x=s \\ 0, & x \neq s \end{cases}$

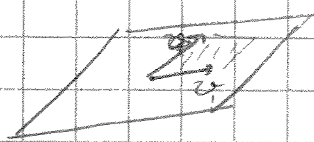
as the free vectorspace over S.

Elements of $F(S)$ are finite sums $\sum_{i=1}^k a_i \chi_{\{s_i\}}$. (a function)

Convention: We denote $\sum_{i=1}^k a_i \chi_{\{s_i\}}$ as $\sum_{i=1}^k a_i s_i$.

Warning: Even if S is a vector space $s + s' = \chi_{\{s\}} + \chi_{\{s'\}}$
$\neq \chi_{\{s+s'\}}$ in $F(S)$.

Exterior product $\Lambda_k V$.



Let $\varphi \in Alt^k(V)$.
Then $\varphi(v_1, \dots, v_k)$ depends only the "area spanned" by $v_1, \dots, v_k$. i.e.

Q: How to formalize this?

Idea 1: Let $\sim$ be equ. relation on V^k s.t.
$(v_1 \dots v_k) \sim (w_1 \dots w_k) \iff \varphi(v_1, \dots, v_k) = \varphi(w_1, \dots, w_k)$.

Then we can define $a\underset{\text{equ. class}}{[(v_1, \dots, v_k)]} = [(av_1, v_2, \dots, v_k)]$.

but how to define addition?

$\leadsto$ It is not clear that $V^k/\sim$ is a vector space.

Idea 2: Let $\mathcal{F} = \operatorname{span}\{\chi_{\{\underline{v}\}} : V^k \to \mathbb{R} \;:\; \underline{v} \in V^k\} \subset \{\text{all functions } V^k \to \mathbb{R}\}$

Elements of $\mathcal{F}$: $\sum_{i=1}^{k} a_i \chi_{\{\underline{v}_i\}}$

We set equ. relation $\sim$ on $\mathcal{F}$ by

$$\sum_{i=1}^{k} a_i \chi_{\{\underline{v}_i\}} \sim \sum_{j=1}^{m} b_j \chi_{\{\underline{w}_j\}}$$

$$\iff \sum_i a_i \varphi(\underline{v}_i) = \sum_j b_j \varphi(\underline{w}_j) \quad \forall \varphi \in Alt^k(V)$$

Lemma: (1) $a\left[\sum_i a_i \chi_{\{\underline{v}_i\}}\right] = \left[\sum_i a a_i \chi_{\{\underline{v}_i\}}\right]$ and

(2) $\left[\sum a_i \chi_{\{\underline{v}_i\}}\right] + \left[\sum_i b_j \chi_{\{\underline{w}_j\}}\right] = \left[\sum a_i \chi_{\{\underline{v}_i\}} + \sum_i b_j \chi_{\{\underline{w}_j\}}\right]$

are well-defined.

Cor: $\Lambda_k V := \mathcal{F}/\sim$ is a vector space.

Pf of Lemma:

(1) Need to show that for $a \in \mathbb{R}$ and $\sum_i a_i \chi_{\{\underline{v}_i\}} \sim \sum_j b_j \chi_{\{\underline{w}_j\}}$ we have

$$\sum_i a a_i \chi_{\{\underline{v}_i\}} \sim \sum_j a b_j \chi_{\{\underline{w}_j\}}.$$

Let $\varphi \in Alt^k(V)$. Then

$$\sum_i a a_i \varphi(\underline{v}_i) = a \sum_i a_i \varphi(\underline{v}_i) \overset{\text{equivalence}}{=} a \sum_j b_j \varphi(\underline{w}_j)$$

$$= \sum_j a b_j \varphi(\underline{w}_j)$$

So $\left[a \sum_i a_i \varphi(\underline{v}_i)\right] = \left[a \sum_j b_j \varphi(\underline{w}_j)\right]$ is well-defined.

(2) Let $\sum_{i=1}^{n_1} a_i \chi_{\{\underline{v}_i\}} \sim \sum_{k=1}^{n_2} a'_k \chi_{\{\underline{v}'_k\}}$ and $\sum_{j=1}^{m_1} b_j \chi_{\{\underline{w}_j\}} \sim \sum_{l=1}^{m_2} b'_l \chi_{\{\underline{w}'_l\}}$

Since, for all $\varphi \in Alt^k(V)$,

$$\sum_i a_i \varphi(\underline{v}_i) + \sum_j b_j \varphi(\underline{w}_j) = \sum_k a'_k \varphi(\underline{v}'_k) + \sum_l b'_l \varphi(\underline{w}'_l)$$

we have that $\left[\sum_i a_i \varphi(\underline{v}_i) + \sum_j b_j \varphi(\underline{w}_j)\right] = \left[\sum a'_k \varphi(\underline{v}'_k) + \sum b'_l \varphi(\underline{w}'_l)\right]$

Thus the addition is well-defined $\square$

Notation: $v_1 \wedge \cdots \wedge v_k = \left[\chi_{\{(v_1, \dots, v_k)\}}\right]$

and

$$\sum_{I=(i_1 \dots i_k)} a_I v_{i_1} \wedge \cdots \wedge v_{i_k} = \left[\sum_I a_I \chi_{\{(v_{i_1}, \dots, v_{i_k})\}}\right]$$

Theorem: Let $\varphi \in Alt^k(V)$. There exists a unique linear map $\underline{\varphi}: \Lambda_k V \to \mathbb{R}$ so that

$$\varphi(v_1, \dots, v_k) = \underline{\varphi}(v_1 \wedge \cdots \wedge v_k)$$

for all $v_1, \dots, v_k \in V$.

$$\begin{tikzcd} V^k \arrow[d, "\varphi"'] \arrow[dr, "\wedge"] & \\ \mathbb{R} & \Lambda_k V \arrow[l, "\underline{\varphi}"] \end{tikzcd}$$

Pf: Let $\sum_i a_i \chi_{\{\underline{v}_i\}} \sim \sum_j b_j \chi_{\{\underline{w}_j\}}$

Then, by def.,

$$\sum_i a_i \varphi(\underline{v}_i) = \sum_j b_j \varphi(\underline{w}_j)$$

Thus the map $\underline{\varphi} : \Lambda_k V \to \mathbb{R}$,

$$\underline{\varphi}\Big(\underbrace{\textstyle\sum_i a_i v_{i_1} \wedge \dots \wedge v_{i_k}}\Big) = \sum_i a_i \varphi(v_{i_1}, \dots, v_{i_k})$$
$$= \sum_i a_i [\chi_{\{(v_{i_1}, \dots, v_{i_k})\}}]$$
$$= \Big[\sum_i a_i \chi_{\{(v_{i_1}, \dots, v_{i_k})\}}\Big]$$

is well-defined.

Since $\underline{\varphi}\left(\sum a_i v_{i_1} \wedge \dots \wedge v_{i_k}\right) = \sum_i a_i \varphi(v_{i_1}, \dots, v_{i_k})$
$$= \sum_i a_i \underline{\varphi}(v_{i_1} \wedge \dots \wedge v_{i_k}),$$
$\underline{\varphi}$ is linear. $\square$

Thm: $Alt^k(V) \to (\Lambda_k V)^*$, $\varphi \mapsto \underline{\varphi}$, is an isomorphism

Pf: Denote $\Phi : Alt^k(V) \to (\Lambda_k V)^*$, $\varphi \mapsto \underline{\varphi}$
and $\Psi : (\Lambda_k V)^* \to Alt^k(V)$, $f \mapsto f \circ \wedge$
where $\wedge : V^k \to \Lambda_k V$ is the map $(v_1, \dots, v_k) \mapsto v_1 \wedge \dots \wedge v_k$.

Then $\underbrace{(\Psi \circ \Phi)(\overset{\in Alt^k(V)}{\varphi})}_{\in Alt^k(V)}(v_1, \dots, v_k) = \Psi(\Phi(\varphi))(v_1, \dots, v_k)$

$$= \Psi(\underline{\varphi})(v_1, \dots, v_k) = (\underline{\varphi} \circ \wedge)(v_1, \dots, v_k)$$

$$= \underline{\varphi}(v_1 \wedge \dots \wedge v_k) = \varphi(v_1, \dots, v_k)$$

So $(\Psi \circ \Phi)(\varphi) = \varphi$. So $\Psi \circ \Phi = id$.

On the other hand, $v_1 \wedge \dots \wedge (a v + v') \wedge \dots \wedge v_k$
$$= a v_1 \wedge \dots \wedge v \wedge \dots \wedge v_k + v_1 \wedge \dots \wedge v' \wedge \dots \wedge v_k$$
(indeed, $\varphi(v_1, \dots, av + v', \dots, v_k) = a\varphi(v_1, \dots, v, \dots, v_k)$
$$+ \varphi(v_1, \dots, v', \dots, v_k)$$
for all $\varphi \in Alt^k(V)$)

Thus $f \circ \wedge \in Alt^k(V)$ for all $f \in (\Lambda_k V)^*$.

Moreover, for $f \in (\Lambda_k V)^*$,

$$(\Phi \circ \Psi)(f)\left(\sum_I a_I v_{i_1} \wedge \dots \wedge v_{i_k}\right) = \Phi(\Psi(f))\left(\sum_I a_I v_{i_1} \wedge \dots \wedge v_{i_k}\right)$$

$$= \sum_I a_I \Psi(f)(v_{i_1}, \dots, v_{i_k}) = \sum_I a_I f(v_{i_1} \wedge \dots \wedge v_{i_k})$$

$$\overset{f \in (\Lambda_k V)^*}{=} f\left(\sum_I a_I v_{i_1} \wedge \dots \wedge v_{i_k}\right) \qquad (I = (i_1 \dots i_k))$$

Thus $\Phi \circ \Psi(f) = f$ for all $f \in (\Lambda_k V)^*$.
Hence $\Phi \circ \Psi = id$.

Remains to show linearity of Ψ, say
Let $f, g \in (\Lambda_k V)^*$ and $a \in \mathbb{R}$. Then

$$\Psi(af+g)(v_1, \dots, v_k) = (af+g)(v_1 \wedge \dots \wedge v_k)$$
$$= a\, f(v_1 \wedge \dots \wedge v_k) + g(v_1 \wedge \dots \wedge v_k)$$
$$= a\, \Psi(f)(v_1, \dots, v_k) + \Psi(g)(v_1, \dots, v_k).$$

So $\Psi(af+g) = a\Psi(f) + \Psi(g)$. $\square$

Rmk: Proof gives that $\wedge : V^k \to \Lambda_k V$ is alternating and k-linear.

Observations:

(1) Let $f : W \to V$ be a linear map.
Then $f_* = \Lambda_k f : \Lambda_k W \to \Lambda_k V$
$$\Lambda_k f\left(\sum_I a_I v_{i_1} \wedge \dots \wedge v_{i_k}\right) = \sum_I a_I f(v_{i_1}) \wedge \dots \wedge f(v_{i_k})$$
is well-defined and linear.

(2) Let $(e_1, \dots, e_n)$ be a basis of V.
Then $(e_{i_1} \wedge \dots \wedge e_{i_k} : 1 \le i_1 < \dots < i_k \le n)$ is a basis of $\Lambda_k V$.
"Pf":

- $v_1 \wedge \dots \wedge v_k = \sum_{i_1 \dots i_k} v_{i_1} \dots v_{i_k}\, e_{i_1} \wedge \dots \wedge e_{i_k}$

- $\sum_I a_I e_{i_1} \wedge \dots \wedge e_{i_k} = 0 \iff \sum_I a_I \varphi(e_{i_1}, \dots, e_{i_k}) = 0 \quad \forall \varphi \in Alt^k(V)$

$\Rightarrow a_I = 0$ for all I. $\square$