IDF Exercise set 12 Solutions (PP)

1. (U, φ), $\varphi = (\varphi_1 \dots \varphi_n)$ a chart.

Claim 1: $\varphi^* dx_i = d\varphi_i \quad \forall i$

Pf: Since $(\varphi^{-1})^* \varphi_i = \varphi_i \circ \varphi = x_i$,
where $x = (x_1 \dots x_n)$ is the identity on φU,
we have by definition of exterior derivative
$$d\varphi_i = \varphi^*(d((\varphi^{-1})^*\varphi_i)) = \varphi^* d x_i$$ □

Claim 2: $((d\varphi_i)_p)_{i=1}^n$ is a basis of $T_p^* M \quad \forall p \in U$.

Pf 1: $(dx_1)_{\varphi(p)}, \dots, (dx_n)_{\varphi(p)}$ is a basis of $T^*_{\varphi(p)} \mathbb{R}^n$
and $\varphi^* : T^*_{\varphi(p)} \mathbb{R}^n \to T_p^* M$ is an isomorphism
(it has an inverse). Thus $(d\varphi_i)_p$ is a basis by Claim 1. □

Pf 2: Since $D\varphi \left(\frac{\partial}{\partial \varphi_k}\right)_p u = \left(\frac{\partial}{\partial \varphi_k}\right)_p (u \circ \varphi) = \frac{\partial}{\partial x_k} (u \circ \varphi \circ \varphi^{-1})(\varphi(p))$
$= \left(\frac{\partial}{\partial x_k} u\right)_{\varphi(p)} \quad \forall u \in C^\infty(\varphi(p))$,
we have
$$(d\varphi_i)\left(\frac{\partial}{\partial \varphi_k}\right)_p = \varphi^* dx_i \left(\frac{\partial}{\partial \varphi_k}\right)_p = dx_i \left(D\varphi \left(\frac{\partial}{\partial \varphi_k}\right)_p \right)$$
$$= dx_i \left(\frac{\partial}{\partial x_k}\right) = \begin{cases} 1, & i = k \\ 0, & \text{otherwise.} \end{cases}$$
Thus $(d\varphi_i)_p$ is a dual basis of $\left(\frac{\partial}{\partial \varphi_k}\right)$. □

Claim 3: $(d\varphi_I)_p$ is a basis of $Alt^k(T_p M)$
where $d\varphi_I = d\varphi_{i_1} \wedge \dots \wedge d\varphi_{i_k}$, $I = (i_1 \dots i_k)$
and $1 \le i_1 < \dots < i_k \le n$.

Pf: Since $(d\varphi_i)_p$ is a basis, the claim follows
from Corollary on page f in Multilinear algebra section
□

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(i) Let $\omega = \sum_I f_I \, dt \wedge dx_I + \sum_J g_J \, dx_J \in \Omega^{k+1}(B^n \times \mathbb{R})$

Then $\omega\left(\frac{\partial}{\partial t}, v_1 \dots v_k\right) = \sum_I f_I \, dt \wedge dx_I \left(\frac{\partial}{\partial t}, v_1 \dots v_k\right)$

since $dx_J\left(\frac{\partial}{\partial t}, v_1 \dots v_k\right) = 0$ for all $(v_1 \dots v_k)$.

Thus

$$(\hat{S}_t \omega)_x (v_1 \dots v_k) = \sum_I \left(\int_0^1 f_I(x,t) \, d\alpha'(t)\right) dx_I (v_1 \dots v_k)$$

$$= \int_0^1 \sum_I f_I(x,t) \, dx_I (v_1 \dots v_k) \, d\alpha'(t)$$

$$= \int_0^1 \sum_I f_I(x,t) \, dt\left(\frac{\partial}{\partial t}\right) dx_I (v_1 \dots v_k) \, d\alpha'(t)$$

$$= \int_0^1 \sum_I f_I(x,t) (dt \wedge dx_I)\left(\frac{\partial}{\partial t}, v_1 \dots v_k\right) d\alpha'(t)$$

$$= \int_0^1 \omega_{(x,t)}\left(\frac{\partial}{\partial t}, v_1 \dots v_k\right) d\alpha'(t).$$

(ii) Let (U, φ) be a chart on M. Then

$$\varphi^* \hat{S}_t\left(((\varphi \times id)^{-1})^*(\omega)\right)_p (v_1 \dots v_k) = \hat{S}_t\left(((\varphi \times id)^{-1})^*(\omega)\right)_{\varphi(p)} \left((D\varphi) v_1, \dots (D\varphi) v_k\right)$$

$$= \int_0^1 \left(((\varphi \times id)^{-1})^*(\omega)\right)_{(\varphi(p), t)} \left(\frac{\partial}{\partial t}, (D\varphi) v_1, \dots (D\varphi) v_k\right) d\alpha'(t)$$

$$= \int_0^1 \omega_{(p,t)}\left(D(\varphi \times id)^{-1} \frac{\partial}{\partial t}, D(\varphi \times id)^{-1} (D\varphi) v_1, \dots D(\varphi \times id)^{-1} (D\varphi) v_k\right) d\alpha'(t)$$

$$= \int_0^1 \omega_{(p,t)}\left(\frac{\partial}{\partial t}, v_1, \dots v_k\right) d\alpha'(t).$$

(iii) We define $\hat{S}_t^M : \Omega^{k+1}(M \times \mathbb{R}) \to \Omega^k(M)$

so that $(\hat{S}_t^M \omega)|U = \varphi^* \circ \hat{S}_t \circ ((\varphi \times id)^{-1})^*(\omega)$

By (ii) the form $\hat{S}_t^M \omega$ is well-defined and

hence $\hat{S}_t^M$ is well-defined.

By the proof of Poincaré's lemma,

$$i_1^* - i_0^* = d\hat{S}_k + \hat{S}_{k+1} d$$

where i_j is the inclusion $x \mapsto (x, j)$.

Thus, on chart (U, φ),

$$d\hat{S}_k^M + \hat{S}_{k+1}^M d = d\left(\varphi^* \hat{S}_k ((\varphi \times id)^{-1})^*\right) + \varphi^* \hat{S}_{k+1}((\varphi \times id)^{-1})^* d$$

$$= \varphi^* (d\hat{S}_k + \hat{S}_{k+1} d)((\varphi \times id)^{-1})^*$$

$$= \varphi^* (i_1^* - i_0^*)((\varphi \times id)^{-1})^* = (i_1^U)^* - (i_0^U)^*$$

where $i_j^U : U \to U \times \mathbb{R}$ $x \mapsto (x, j)$.

Thus $d\hat{S}_k^M + \hat{S}_{k+1}^M d = i_1^* - i_0^*$. where $i_\ell : M \to M \times \mathbb{R}$, $x \mapsto (x, \ell)$

(iv) Let $F : M \times \mathbb{R} \to N$ be a smooth homotopy from g to f.

Then

$$f^* - g^* = (F \circ i_1)^* - (F \circ i_0)^* = i_1^* \circ F^* - i_0^* \circ F^*$$

$$= (d\hat{S}_k^M + \hat{S}_{k+1}^M d) \circ F^*$$

$$= d \circ (\hat{S}_k^M \circ F^*) + (\hat{S}_{k+1}^M \circ F^*) d.$$

We define $S_k = \hat{S}_k^M \circ F^*$.

Cor: If $f, g : M \to N$ are smoothly homotopic smooth maps then $f^* = g^* : H^k(N) \to H^k(M)$.

Pf: Let $[\omega] \in H^k(N)$. Then

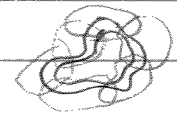
$$[f^*\omega] = [g^*\omega + dS_k\omega + S_{k+1} d\omega]$$

$$= [g^*\omega + dS_k\omega] = [g^*\omega] \qquad \square$$

3. M and N smooth m- and n-manifolds, respectively.
$f : M \to N$ continuous map.

(i) Let $\mathcal{A}' = \{(V_j, \psi_j) : j \in J\}$ be an atlas of N s.t. $\psi_j V_j = B^n$.
By continuity of f, $\exists$ atlas $\{(U'_p, \varphi_p) : p \in M\}$ s.t. $\forall p \in M$ $\exists j \in J$ s.t. $f U'_p \subset V_j$. May assume $\varphi_p(p) = 0$ and $\varphi'_p U'_p = B^m$, $\overline{U'_p}$ compact.
Using partition of unity, $\exists$ a countable atlas $\{(U'_{p_i}, \varphi_{p_i}) : i \geq 0\} \subset \{(U'_p, \varphi_p) : p \in M\}$ that is locally finite. (Use partition of unity to $\{U'_p\}_{p \in M}$.)

By local finiteness of the atlas, $\forall i \geq 0$ $\exists r_i \in (0,1)$ s.t.
$$\bigcup_{i \geq 0} \varphi_{p_i}^{-1} B^m(0, r_i) = M.$$



We take $K_i = \varphi_{p_i}^{-1} \overline{B}^m(0, r_i)$, $W_i = \varphi_{p_i}^{-1}\left(B^m\left(0, \frac{r_i + 1}{2}\right)\right)$
and $(U_i, \varphi_i) = (U_{p_i}, \varphi_{p_i})$. ∎

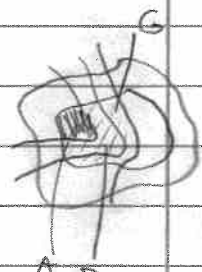
(ii) Let $g_{-1} = f$ and $W_{-1} = \emptyset$. Let $j \in J$ s.t. $f U_0 \subset V_j$
and $h_0 : \varphi_0 U_0 \to \psi_j V_j$ be the map $h_0 = \psi \circ f \circ \varphi_0^{-1}$.
Let $\varepsilon : \varphi_0 W_0 \to [0, \infty)$ be the function
$$\varepsilon(x) = \min\{\operatorname{dist}(h_0(x), \partial \psi_j V_j), \operatorname{dist}(x, \partial \varphi_0 W_0)\}.$$
Then $\exists$ smooth map $\tilde{h}_0 : \varphi_0 U_0 \to \psi_j V_j$ homotopic to h_0 s.t. $|\tilde{h}_0(x) - h_0(x)| < \varepsilon(x)$.
Let $x \in \partial \varphi_0 W_0$ and $x_i \to x$ in $\varphi_0 W_0$.
Then
$$|\tilde{h}_0(x_i) - h_0(x)| \leq |\tilde{h}_0(x_i) - h_0(x_i)| + |h_0(x_i) - h_0(x)| \leq \varepsilon(x_i) + |h_0(x_i) - h_0(x)| \xrightarrow[i \to \infty]{} 0.$$
Hence we may extend $\tilde{h}_0$ to a continuous map $\tilde{h}_0 : \varphi_0 U_0 \to \psi_j V_j$ s.t. $\tilde{h}_0 = h_0$ on $\varphi_0 U_0 \setminus \varphi_0 W_0$.

Define $g_0 : M \to N$ by $g_0 | U_0 = \psi_j^{-1} \circ \tilde{h}_0 \circ \varphi_0$
and $g_0 | M \setminus U_0 = f$.
Then g_0 is continuous and $g_0 | W_0$ is smooth.
Moreover, g_0 is homotopic to f rel $M \setminus W_0$.

Induction assumption:

Suppose now that we have constructed maps $g_0, \dots, g_{k-1} : M \to N$ s.t. $g_j | M \setminus W_j = g_{j-1} | M \setminus W_j$, g_j is homotopic to g_{j-1} rel $M \setminus W_j$ and g_j is smooth in a neighborhood of $\bigcup_{i=0}^{j} K_i$.

Let $j \in J$ s.t. $fU_k \subset V_j$. Denote $h_k = \psi_j \circ g_{k-1} \circ \varphi_k^{-1} : \varphi_k U_k \to \psi_j V_j$.
Let $A = \varphi_k (U_k \cap \bigcup_{i=0}^{k} K_i)$. Then, by assumption, h_k is smooth in a neighborhood, say G, of A. Fix $\underset{\text{open}}{D}$ s.t. $A \cap \varphi_k U \subset D \subset \overline{D} \subset G$.
Let $\varepsilon : \varphi_k U_k \to (0, \infty)$ be the function
$\varepsilon(x) = \min \{ \operatorname{dist}(h_k(x), B^n), \operatorname{dist}(x, \partial \varphi_k W_k) \}$.



Then $\exists$ a smooth map $\tilde{h}_k : \varphi_k W_k \to \psi_j V_j$ s.t.
$\tilde{h}_k$ is smooth in $\varphi_k W_k$, $\tilde{h}_k$ is homotopic to h_k,
$\tilde{h}_k | (\overline{D} \cap \varphi_k W_k) = h_k | \overline{D} \cap \varphi_k W_k$.

Again, we can extend $\tilde{h}_k$ to a map $\tilde{h}_k : \varphi_k U_k \to \psi_j V_j$ (as in the case $k=0$).

Then the map $g_k : M \to N$, $g_k | U_k = \psi_j^{-1} \circ \tilde{h}_k \circ \varphi_k$, $g_k | M \setminus U_k = g_{k-1} | M \setminus U_k$ satisfies the conditions of the induction step.

(iii) Given the sequence in (ii)
we define $G : M \times [-1, \infty) \to N$ s.t.
$G | M \times [k, k+1]$ is a homotopy from g_k to g_{k+1}
s.t. $G(x,t) = g_k(x)$ for $t \in [k, k+1]$ if $x \in M \setminus W_{k+1}$.
Since $\{U_i\}$ is locally finite
$\forall x \in M \ \exists k_0$ s.t. $G(x,t) = G(x, k_0)$ for $t \geq k_0$.

Thus $F : M \times [0,1] \to N$ where $F(x,t) = G(x, \frac{2}{\pi} \arctan t)$
and $F(x,1) = \lim_{t \to \infty} G(x,t)$, is continuous.

Moreover $F \circ i_0 = f$ and $F \circ i_1$ is a smooth map.
$\square$

4. Claim: M smooth n-manifold has a positive atlas
$\iff$ $\exists$ an n-form $\omega \in \Omega^n(M)$ s.t. $\omega_p \neq 0$ $\forall p \in M$

Pf: Suppose M has a positive atlas $\{(U_i, \varphi_i) : i \in I\}$.
Let $\{\psi_i\}$ be a partition of unity w.r.t. $\{U_i\}$.
Define $\omega = \sum_i \psi_i \varphi_i^* dx_1 \wedge \dots \wedge dx_n$

Let $p \in M$ and fix $j \in I$ s.t. $\psi_j(p) > 0$.
Suppose $i \in I$ s.t. $p \in U_i$. Then in $U_i \cap U_j$,

$$\varphi_i^*(dx_1 \wedge \dots \wedge dx_n) = (\varphi_j^{-1} \circ \varphi_j)^* \varphi_i^*(dx_1 \wedge \dots \wedge dx_n)$$
$$= \varphi_j^*((\varphi_i \circ \varphi_j^{-1})^* dx_1 \wedge \dots \wedge dx_n)$$
$$= \varphi_j^*(J_{\varphi_i \circ \varphi_j^{-1}} dx_1 \wedge \dots \wedge dx_n)$$

$$= J_{\varphi_i \circ \varphi_j^{-1}} \circ \varphi_j \; \varphi_j^* dx_1 \wedge \dots \wedge dx_n$$

Since $J_{\varphi_i \circ \varphi_j^{-1}} > 0$ and $\{\psi_i\}$ is a partition of unity

$$\omega_p = \left(\sum_i \psi_i(p) J_{\varphi_i \circ \varphi_j^{-1}}\right)(\varphi_j^* dx_1 \wedge \dots \wedge dx_n)_p \neq 0.$$

$\square$

Suppose $\omega \in \Omega^n(M)$ $\omega_p \neq 0$ $\forall p \in M$.
Let $\mathcal{A}$ be an atlas on M. s.t. charts are connected.
Let $(U, \varphi) \in \mathcal{A}$. Then $(\varphi^{-1})^* \omega = f_\varphi dx_1 \wedge \dots \wedge dx_n$
where $f_\varphi(x) \neq 0$ $\forall x \in \varphi U$.
Define $\mathcal{A}_+ = \{(U, \varphi) : f_\varphi > 0\}$
If $(U, \varphi) \in \mathcal{A} \setminus \mathcal{A}_+$ (with U connected), then $(U, \varphi \circ \sigma) \in \mathcal{A}_+$
where $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ $(x_1 \dots x_n) \mapsto (x_1 \dots x_{n-1}, -x_n)$.
Thus $\mathcal{A}_+$ is an atlas.

$\square$

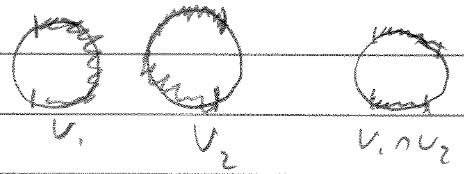
Let $n>0$.

5. (i) Since $i: S^n \to \mathbb{R}^{n+1}\setminus\{0\}$ inclusion
and $\pi: \mathbb{R}^{n+1}\setminus\{0\} \to S^n$ $\quad x \mapsto \frac{x}{|x|}$
give an homotopy equivalence, we have
$$H^k(S^n) \cong H^k(\mathbb{R}^{n+1}\setminus\{0\}) \quad \forall k.$$
Thus $H^k(S^n) \cong \begin{cases} \mathbb{R}, & k=0,n \\ 0, & \text{otherwise} \end{cases}$

For $n=0$, $S^0=\{\pm 1\}$. Thus S^0 is homotopy equivalent to $B^2(e_1,\frac{1}{2}) \cup B^2(-e_1,\frac{1}{2})$
Hence $H^k(S^0) \cong H^k(B^2(e_1,\frac{1}{2})) \oplus H^k(B^2(-e_1,\frac{1}{2}))$
$\cong \begin{cases} \mathbb{R}^2, & k=0 \\ 0, & \text{otherwise} \end{cases}$

[Here we take the stand that a point is a zero dimensional manifold with zero dimension tangent space $\{0\}$. Then k-forms $\{\text{point}\} \to Alt^k(T\{\text{point}\})$ are zero functions]

(ii) Let $S^1=\{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\}$ and
$U_1=\{(x,y)\in S^1 : x>-\frac{1}{2}\}$
$U_2=\{(x,y)\in S^1 : x<\frac{1}{2}\}$



Then $S^1=U_1\cup U_2$ and $U_1\cap U_2$ consists of two components both homeomorphic to interval $(-\frac{1}{2},\frac{1}{2})$.

Since U_1 and U_2 are homeomorphic to an interval, we have that $S^2\times U_i$, $i=1,2$, is homotopy equivalent to S^2. Hence $H^k(S^2\times U_i)\cong H^k(S^2)\cong\begin{cases}\mathbb{R}, & k=0,2\\ 0, & \text{otherwise}\end{cases}$
Moreover, $H^k(S^2\times U_1 \cap S^2\times U_2) = H^k(S^2\times(U_1\cap U_2))$
$\cong H^k(S^2)\oplus H^k(S^2) \cong \begin{cases}\mathbb{R}^2, & k=0,2\\ 0, & \text{otherwise}\end{cases}$

Thus by Mayer-Vietoris,

$$\to H^{k-1}(S^2\times(U_1\cap U_2)) \xrightarrow{\partial_{k-1}^*} H^k(S^2\times S^1) \to H^k(S^2\times U_1)\oplus H^k(S^2\times U_2)$$

$$\to H^k(S^2\times(U_1\cap U_2)) \xrightarrow{\partial_k^*} H^{k+1}(S^2\times S^1) \to$$

reads for $k=1$ as

$$0 \to \underbrace{H^0(S^2\times S^1)}_{\cong\mathbb{R}} \xrightarrow{I_1^*} \underbrace{H^0(S^2\times U_1)\oplus H^0(S^2\times U_2)}_{\cong\mathbb{R}\oplus\mathbb{R}} \xrightarrow{J_1^*} \underbrace{H^0(S^2\times(U_1\cap U_2))}_{\cong\mathbb{R}^2}$$

$$\xrightarrow{\partial_1^*} H^1(S^2\times S^1) \to 0$$

Thus $\ker J_1^* = \operatorname{Im} I_1^* \cong \mathbb{R}$ and $H^1(S^2\times S^1) = \operatorname{Im}\partial_1^*$

Since $\dim \operatorname{Im}\partial_1^* = 2 - \dim\ker\partial_1^* = 2 - \dim\operatorname{Im} J_1^*$

$= 2-1 = 1,$

we have $\dim H^1(S^2\times S^1) = 1.$

For $k=2$, the sequence reads as

$$0 \xrightarrow{\partial_1^*} H^2(S^2\times S^1) \xrightarrow{I_2^*} \underbrace{H^2(S^2\times U_1)\oplus H^2(S^2\times U_2)}_{\cong\mathbb{R}\oplus\mathbb{R}}$$

$$\xrightarrow{J_2^*} \underbrace{H^2(S^2\times(U_1\cap U_2))}_{\cong\mathbb{R}\oplus\mathbb{R}} \xrightarrow{\partial_2^*} H^3(S^2\times S^1) \xrightarrow{I_3^*} 0$$

Thus $\ker I_2^* = 0$ and $\operatorname{Im}\partial_2^* = H^3(S^2\times S^1)$.

Since $\ker\partial_2^* = \operatorname{Im} J_2^*$ and $\ker J_2^* = \operatorname{Im} I_2^*$

we have $2 = \dim\ker J_2^* + \dim\operatorname{Im} J_2^*$

$= \dim\operatorname{Im} I_2^* + \dim\ker\partial_2^*$

$= \dim\operatorname{Im} I_2^* + 2 - \dim\operatorname{Im}\partial_2^*$

$= \dim H^2(S^2\times S^1) + \dim H^3(S^2\times S^1)$

At this point we cannot decide dimensions without some extra information (If we would know that $H^3(S^2\times S^1)\cong\mathbb{R}$ we would be done.)

Observe that $\Omega^k(S^2\times S^1) = \{0\}$ for $k>3$ so $H^k(S^2\times S^1) = 0$ for $k>3$

Method 1:

Let W_1 and W_2 be components of $V_1 \cap V_2$.
and $k_\ell : S^2 \times W_\ell \to S^2 \times (V_1 \cap V_2)$ be inclusion.

Let $j_\ell : S^2 \times (V_1 \cap V_2) \to S^2 \times V_\ell$ be the inclusion, in the definition of J. Then

$j_\ell \circ k_m$ is a homotopy equivalence

Thus $(j_\ell \circ k_m)^*$ is an iso and j_ℓ^* is injective.

Since $S^2 \times V_\ell$ is homotopy equivalent to S^2,
and $\dim H^2(S^2) = 1$, we may fix $[\omega_\ell] \in H^2(S^2 \times W_\ell)$
s.t. $[\omega_\ell]$ generates $H^2(S^2 \times W_\ell)$.

Thus $J(a[\omega_1], b[\omega_2]) = a\, j_1^*[\omega_1] - b\, j_2^*[\omega_2]$

and $k_\ell^* J(a[\omega_1], b[\omega_2]) = a(j_1 \circ k_\ell)^*[\omega_1] - b(j_2 \circ k_\ell)^*[\omega_2]$

$= a\lambda[\tilde{\omega}_\ell] - b\mu[\tilde{\omega}_\ell] = (a\lambda - b\mu)[\tilde{\omega}_\ell]$

where $[\tilde{\omega}_\ell]$ is some fixed generator of $H^2(S^2 \times W_\ell) \cong \mathbb{R}$
and $\lambda \neq 0 \neq \mu$.

Thus $(k_1^* \oplus k_2^*) \circ J : H^2(S^2 \times V_1) \oplus H^2(S^2 \times V_2) \to H^2(S^2 \times W_1) \oplus H^2(S^2 \times W_2)$
has the formula $(a[\omega_1], b[\omega_2]) \mapsto (a\lambda - b\mu)([\tilde{\omega}_1], [\tilde{\omega}_2])$

Thus $\dim \operatorname{Im} (k_1^* \oplus k_2^*) \circ J = 1$

Since $k_1^* \oplus k_2^*$ is an isom, $\dim \operatorname{Im} J = 1$.

We observe that

$\dim H^3(S^2 \times S) = \dim \operatorname{Im} \partial_2^* = 2 - \dim \operatorname{Im} J_2^*$

$= 2 - 1 = 1$ $\square$

Method 2:

Consider $U_1 = \{(x,y,z) \in S^2 : x > -\frac{1}{2}\}$
$U_2 = \{(x,y,z) \in S^2 : x < \frac{1}{2}\}$

Then $U_1 \simeq D^2 \simeq U_2$ $\quad U_1 \cap U_2 \simeq (-1,1) \times S^1$
and $U_1 \times S^1 \simeq U_2 \times S^1 \simeq S^1$. (homotopy equivalences)

By Mayer-Vietoris

$$0 \to H^0(S^2 \times S^1) \xrightarrow{I_0^*} H^0(U_1 \times S^1) \oplus H^0(U_2 \times S^1) \xrightarrow{J_0^*} H^0((U_1 \cap U_2) \times S^1)$$

$$\xrightarrow{\partial_0^*} \underbrace{H^1(S^2 \times S^1)}^{\cong \mathbb{R}} \xrightarrow{I_1^*} H^1(U_1 \times S^1) \oplus H^1(U_2 \times S^1) \xrightarrow{J_1^*} H^1((U_1 \cap U_2) \times S^1)$$

$$\xrightarrow{\partial_1^*} H^2(S^2 \times S^1) \xrightarrow{I_2^*} \underbrace{H^2(U_1 \times S^1) \oplus H^2(U_2 \times S^1)}_{=0} \xrightarrow{J_2^*} H^2((U_1 \cap U_2) \times S^1) -$$

$$\xrightarrow{\partial_2^*} H^3(S^2 \times S^1) \xrightarrow{I_3^*} \underbrace{H^3(U_1 \times S^1) \oplus H^3(U_2 \times S^1)}_{=0} \xrightarrow{J_3^*} \underbrace{H^3((U_1 \cap U_2) \times S^1)}_{=0} \to 0$$

(Here we use the facts that $(U_1 \cap U_2) \times S^1 \simeq S^1 \times S^1$
$\Omega^k(S^2 \times S^1) = \{0\}$ for $k > 3$, $\Omega^k(S^1 \times S^1) = \{0\}$ for $k > 2$
and $H^k(U_1 \times S^1) \cong H^k(U_2 \times S^1) \cong H^k(S^1) = 0$ for $k > 1$.)

We observe first that:
$\ker J_0^* = \operatorname{Im} I_0^* \cong \mathbb{R}$. Thus $\operatorname{Im} J_0^* \cong \mathbb{R} \cong H^0((U_1 \cap U_2) \times S^1)$.
Hence $\ker \partial_0^* = H^0((U_1 \cap U_2) \times S^1)$ and $\partial_0^* = 0$
Thus $\ker J_1^* = \operatorname{Im} I_1^* \cong H^1(S^2 \times S^1) \cong \mathbb{R}$
Hence $\dim H^2(S^2 \times S^1) = \dim \operatorname{Im} \partial_1^* = \dim H^1((U_1 \cap U_2) \times S^1) - 1$.

We show $\dim H^1(S^1 \times S^1) \geq 2$. Fix $q_0 \in S^1$.
Let $i_1 : S^1 \to S^1 \times S^1$ be the map $p \mapsto (p, q_0)$
and $i_2 : S^1 \to S^1 \times S^1$ the map $p \mapsto (q_0, p)$.
Let also $\pi_\ell : S^1 \times S^1 \to S^1$ be the map $(p_1, p_2) \mapsto p_\ell$.
Then $\pi_\ell \circ i_\ell = id : S^1 \circlearrowleft$ and $(\pi_2 \circ i_1)^* = 0 : \Omega^1(S^1) \to \Omega^1(S^1)$
similarly as $(\pi_1 \circ i_2)^* = 0 : \Omega^1(S^1) \to \Omega^1(S^1)$.
Since $H^1(S^1) \cong \mathbb{R}$ $\exists$ $\omega \in \Omega^1(S^1)$ s.t. $[\omega] \neq 0$.
Then $\omega_\ell = \pi_\ell^* \omega$ satisfies $[\omega_\ell] \neq 0$. Indeed,
$i_\ell^* [\omega_\ell] = [(\pi_\ell \circ i_\ell)^* \omega] = [\omega] \neq 0$.

If $a[\omega_1] + b[\omega_2] = 0$ and $a \neq 0$ then

$[\omega_1] = \frac{b}{a}[\omega_2]$.

However, $0 \neq i_1^*[\omega_1] = \frac{b}{a}[i_1^*\omega_2] = \frac{b}{a}[(\pi_2 \circ i_1)^*\omega] = 0$

↯

This shows that $[\omega_1]$ and $[\omega_2]$ are linearly independent.

Thus $\dim H^1(S^1 \times S^1) \geq 2$.

Since $\dim H^2(S^2 \times S^1) = \dim H^1(S^1 \times S^1) - 1$

$\dim H^2(S^2 \times S^1) \geq 2$

and $\dim H^2(S^2 \times S^1) + \dim H^3(S^2 \times S^1) = 2$

we have that $\dim H^1(S^1 \times S^1) = 2$,

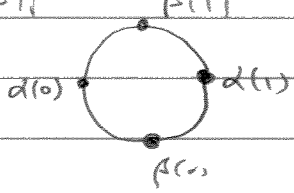
$\dim H^2(S^2 \times S^1) = 1$ and $\dim H^3(S^2 \times S^1) = 1$.

<u>Comment</u> $H^n(N) \cong \mathbb{R}$ for orientable compact n-manifold N

can be used twice in the proof.

<u>6</u> Let $\alpha : [0,1] \to \bar{B}^2$ and $\beta : [0,1] \to \bar{B}^2$

be inj. paths so that

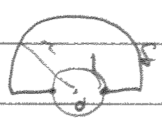


<u>Claim:</u> $\alpha[0,1] \cap \beta[0,1] \neq \emptyset$.

<u>Pf:</u> Let $f : [0,1] \to \mathbb{R}^2 \setminus B^2$

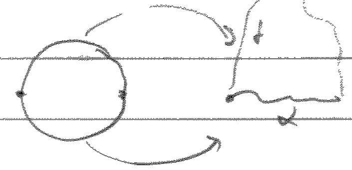
be an injective path so that $f(0) = \alpha(0)$ and $f(1) = \alpha(1)$

and



Let $\gamma : S^1 \to \mathbb{R}^2$ be

$$\gamma(x,y) = \begin{cases} \alpha(\frac{1}{2}(x+1)), & y \leq 0 \\ f(\frac{1}{2}(x+1)), & y \geq 0 \end{cases}$$



Then γ is an embedding

Let $\Sigma = \gamma S^1$. Then by Jordan–Brouwer,

$\mathbb{R}^2 \setminus \Sigma$ has two components U_1 and U_2

where U_1 is bounded and U_2 unbounded.

Moreover $\partial U_1 = \Sigma = \partial U_2$

If either e_2 or $-e_2$ are on $\alpha[0,1]$, then the claim follows. If not, then e_2 and $-e_2$ are in $\mathbb{R}^2 \setminus \Sigma_1$. Let $R = \{t e_2 : t \in (-\infty, -1]\}$ be an infinite ray from $-e_2$ not meeting Σ. Thus $-e_2 \in U_2$.

Let $L = \{t e_2 : t \in [1,2]\}$. Then L is a segment from e_2 contained in $\mathbb{R}^2 \setminus \Sigma_1$.

By construction, $(2+\frac{1}{4})e_2$ is in the same component U_2 of $\mathbb{R}^2 \setminus \Sigma$ as R. Let W_1 and W_2 be the components of $B^2(2e_2, \frac{1}{2}) \setminus$ so that $W_2 \in U_2$. Since $\Sigma_1 = \partial U_2$, we have that $W_1 \not\subset U_2$. Thus, by connectedness, $W_1 \subset U_1$. Since $W_1 \cap L \neq \emptyset$ and L is connected $e_2 \in U_1$. Thus $\beta(0)$ and $\beta(1)$ belong to different components of $\mathbb{R}^2 \setminus \Sigma_1$. Hence $\beta[0,1] \cap \Sigma \neq \emptyset$. Thus $\beta[0,1] \cap \alpha[0,1] \neq \emptyset$.

□