Introduction to differential forms
Problem set 4: model solutions

1. i) In order to show that $\sim$ is an equivalence relation we must show reflexivity, symmetry and transitivity.

<u>Reflexivity</u>: ~~A~~ we must show that $(v_1, \dots, v_n) \sim (v_1, \dots, v_n)$. In this case the map $v_i \mapsto v_i$ is the identity map id_{V}, and $A[id_V, (v_1, \dots, v_n)] = I$ which has a positive determinant.

<u>Symmetry</u>: we must show that $(v_1, \dots, v_n) \sim (w_1, \dots, w_n)$ implies $(w_1, \dots, w_n) \sim (v_1, \dots, v_n)$. But the map $f: v_i \mapsto w_i$ is inverted by $F: w_i \mapsto v_i$, and $A[F, (w_1, \dots, w_n)] = A[f, (v_1, \dots, v_n)]^{-1}$ and if one has a positive determinant so does the other.

<u>Transitivity</u>: Let $f: u_i \mapsto v_i$ ~~ad~~ and $g: v_i \mapsto w_i$ be such that $A[f, (u_1, \dots, u_n)]$ and $A[g, (v_1, \dots, v_n)]$ have positive determinant. Then $A[g \circ f, u_i] = A[g, (v_1, \dots, v_n)] \cdot A[f, (u_1, \dots, u_n)]$ and hence has positive determinant, and $g \circ f: u_i \mapsto w_i$.

<u>Two Elements</u>: two bases will be related by an invertible matrix. ~~If~~ This will have either a positive or negative determinant. If negative then they are not related. Any basis not related to the ~~first~~ second will then have to be related to the first.

1. ii) we must show that if $(v_1, \dots, v_n) \sim (w_1, \dots, w_n)$ then $v_1 \wedge \dots \wedge v_n \sim w_1 \wedge \dots \wedge w_n$ and vice versa.

But $w_i = \sum A[f,(v_1, \dots, v_n)]_{ij} \, v_i$

and so $w_1 \wedge \dots \wedge w_n = f_* v_1 \wedge \dots \wedge f_* w$

and $\cancel{w_1 \wedge \dots \wedge w} = \det A[f,(v_1, \dots, v_n)] \; v_1 \wedge \dots \wedge v_n$

but then $\det A[f,(v_1, \dots, v_n)]$ is positive if and only if f is orientation preserving.

2 i) First we write $X_p = \sum_{i=1}^{n} \frac{x_i}{|X|} e_i$ where $p=(x_1,\dots,x_n)$

Then $\omega(v_1,\dots,v_n) =$

$$\sum_{i=1}^{n} \frac{x_i}{|X|}(-1)^{i-1} dx_i \wedge (dx_1\wedge\dots\wedge d\hat{x}_i\wedge\dots\wedge dx_n)(e_i, v_1,\dots,v_n)$$

$$= \sum_{i=1}^{n} \frac{x_i}{|X|}(-1)^{i-1} dx_1\wedge\dots\wedge d\hat{x}_i\wedge\dots\wedge dx_n(v_1,\dots,v_n).$$

Hence $\omega = \sum_{i=1}^{n} \frac{x_i}{|X|}(-1)^{i-1} dx_1\wedge\dots\wedge d\hat{x}_i\wedge\dots dx$ and is smooth on $\mathbb{R}^n\setminus\{0\}$

ii) First we note that $f^*(\alpha\wedge\beta) = f^*\alpha\wedge f^*\beta$ and $f^*(d\alpha)=d(f^*\alpha)$. Hence $\sigma_+^*(\omega)$ is equal to

$$\sigma_+^*(\omega) = \sum_{i=1}^{n} \frac{\sigma_+^*(x_i)}{\sigma_+^*(|X|)}(-1)^{i-1} d(\sigma_+^*(x_1))\wedge\dots\wedge\widehat{(d\sigma_+^*(x_i))}\wedge\dots d(\sigma_*^*(x_n))$$

$$= \sum_{i=1}^{n} \sigma_+^{i} (-1)^{i-1} d\sigma_+^1\wedge\dots\wedge d\hat{\sigma}_+^i\wedge\dots\wedge d\sigma_+^n$$

Letting y^i denote the standard coordinates of $\mathbb{R}^{n-1}$

$$\sigma_+^*(\omega) = \sum_{i=1}^{n-1} y^i(-1)^{i-1} dy^1\wedge\dots\wedge d\hat{y}_i\wedge\dots\wedge d(\sqrt{1-|y|^2})$$
$$+ \sqrt{1-|y|^2}(-1)^{n-1}dy^1\wedge\dots\wedge dy^{n-1}$$

$$= \sum_{i=1}^{n-1} y^i(-1)^{i-1} dy^1\wedge\dots\wedge d\hat{y}_i\wedge\dots\wedge \sum_{j=1}^{n-1}\frac{-y^j dy^j}{\sqrt{1-|y|^2}}$$
$$+(-1)^{n-1}\sqrt{1-|y|^2}\, dy^1\wedge\dots\wedge dy^{n-1}$$

$$= \sum_{i=1}^{n-1} \frac{-y_i^2}{\sqrt{1-|y|^2}}(-1)^{i-1}(-1)^{(n-1)-i}\, dy^1\wedge\dots\wedge dy^{n-1} + (-1)^{n-1}\sqrt{1-|y|^2}\, dy^1\wedge\dots\wedge dy^{n-1}$$

$$= \left(\frac{1-|y|^2}{\sqrt{1-|y|^2}} (-1)^{n-1} + \frac{|y|^2(-1)^{n-1}}{\sqrt{1-|y|^2}} \right] dy^1 \wedge \dots \wedge dy^{n-1}$$

$$= \frac{1}{\sqrt{1-|y|^2}} (-1)^{n-1} \, dy^1 \wedge \dots \wedge dy^{n-1}$$

A similar calculation yields that

$$\sigma_-^*(\omega) = \frac{1}{\sqrt{1-|y|^2}} (-1)^n \, dy^1 \wedge \dots \wedge dy^{n-1}$$

iii) for $n = 3$ we make the polar substitution

$$\int_{B(0,1)} \sigma_+^*(\omega) = \int_{B(0,1)} \frac{1}{\sqrt{1-|y|^2}} (-1)^{n-1} \, dy^1 \wedge \dots \wedge dy^{n-1}$$

$$= \int_0^{2\pi} \int_0^1 \frac{1}{\sqrt{1-r^2}} \, r \, dr \, d\vartheta$$

$$= 2\pi \left[-\sqrt{1-r^2} \right]_0^1$$

$$= 2\pi$$

similarly

$$\int_{B(0,1)} \sigma_-^*(\omega) = -2\pi$$

3. Let $e_1, \dots, e_k$ ~~by~~ be an ON basis of W. and $e_{k+1}, \dots, e_n$ complete it to an ON basis of V. As such $e_{k+1}, \dots, e_n$ is an ON basis of $W^\perp$. Suppose $e_1, \dots, e_k$ be ordered so that $e_1 \wedge \dots \wedge e_k \sim \xi_W$, and ~~let~~ $e_{k+1}, \dots e_n$ be ordered so that $e_1 \wedge \dots \wedge e_n \sim \xi_V$. Then

$$\xi_W \wedge e_{k+1} \wedge \dots \wedge e_n \sim e_1 \wedge \dots \wedge e_k \wedge \dots \wedge e_n \sim \xi_V.$$

So $e_{k+1} \wedge \dots \wedge e_n$ appropriately scaled is a suitable orientation for $W^\perp$

4. Now we must calculate the respective integrals.

$$\int_P \omega = \int_{\mathbb{R}^k} \varphi^* \omega \quad \text{where } \varphi \text{ is an orientation}$$

preserving affine isomorphism.

Let $\varphi : e_i \mapsto e_i' + p$ then

$$\int_{\mathbb{R}^k} \varphi^* \omega = \int_{\mathbb{R}^k} \underline{\omega}_{\varphi(x)}(\xi)\, dx \,.$$

Now we must show that this is equal to

$$c \int_P \underline{\omega}_x(x, \xi)\, d\mathcal{H}^k(x).$$

To see this we must note that $\mathcal{H}^k$ is invariant under isometry so we can translate ~~by~~ and rotate by the map Φ, which takes $e_i' + p$ to e_i for e_i' a basis of $\mathbb{R}^n$ and e_i the standard ~~basis~~.

Then

$$\int_P \underline{\omega}_x(x, \xi)\, d\mathcal{H}^k(x) = \int_{\Phi(P)} \underline{\omega}_{\Phi^{-1}(y)}(\Phi^{-1}(y), \xi)\, d\mathcal{H}^k(y)$$

$$= \int_{\mathbb{R}^k} \underline{\omega}_{\varphi(y)}(\varphi(y), \xi)\, d\mathcal{H}^k(y)$$

Now we note that $\mathcal{H}^k$ is a haar measure for $\mathbb{R}^k$ and so is equal to a constant times Lebesgue measure. And so we're done.

5 i) Suppose $\dim T_p(f) = n$ then

$Df_p(e_1), \dots, Df_p(e_n)$ are linearly independent and $Df_p(e_1) \wedge \dots \wedge Df_p(e_n) \neq 0$. The reverse implication also holds.

ii) The map f must be injective, but the differential need not be ~~on~~ consider $x \xmapsto{f} x|x|$. with derivative $2|x| e_1 \otimes dx_1$, which is cts. $D_0 f(e_1) = 0$ so $\dim T_0(f) = 0 < 1$.

6 i) Applying the theory of covering spaces, we know that ~~S~~ $\mathbb{R} \xrightarrow{\exp(i\pi 2\cdot)} S^1 \quad t \mapsto e^{i2\pi t}$,

and that $\mathbb{Z}$ acts transitively and totally discontinuously on the fiber of $\exp(2\pi i\,\cdot)$ and hence on $\pi_1(S^1)$
so $S^1 \cong \mathbb{R}/\mathbb{Z}$, while $\pi_1(\mathbb{R}) = 0$.
But $\pi_1(S^1)/\mathbb{Z} = \pi_1(\mathbb{R}) = 0$, so $\pi_1(S^1) = \mathbb{Z}$

ii) Let γ be a curve in $X \times Y$, then

$\gamma = (\gamma_X, \gamma_Y)$ in coordinates.
Now $\gamma \sim \eta$ if and only if

$\gamma_X \sim \eta_X$ and $\gamma_Y \sim \eta_Y$.
To see this let H ~~H_Y~~ be a homotopy from γ to η, then H_X is a homotopy from γ_X to η_X and H_Y is from γ_Y to η_Y.
The converse is given by Homotopies H^1, H^2 from γ_X to η_X and γ_Y to η_Y respectively. Then (H^1, H^2) is a homotopy from γ to η. Thus the spaces are bijective, So, we must show $\gamma_1 \times \eta \cdot \gamma_2 \times \eta\,0 \sim \gamma_1 \cdot \gamma_2 \times \eta$.
But here we take the homotopy from ~~η~~$\cdot\eta\cdot 0$ to η, H and combine it with $(\gamma_1 \cdot \gamma_2, H)$ to yield the desired identification

6. iii) $\mathbb{R}^2$ via $t, s \mapsto \exp(2\pi i t) \times \exp(2\pi i s)$

and $\mathbb{R} \times S^1$ via $t, e^{i\theta} \mapsto e^{2\pi i t} \times e^{i\theta}$.