STOCHASTIC PARTICLE SYSTEMS: EXERCISE 6

1. Consider a reversible measure μ for a strongly continuous contraction semigroup $S(t)$ on $C(X)$ with generator L. Moreover, let $\tilde{L}$ be the L^2 extension of L considered in the lectures. Show that $\tilde{L}$ is self adjoint.

Solution: One simple way to check self adjointness of an operator is to show that the range of $\tilde{L} \pm i$ is the entire Hilbert space (see Reed and Simon volume 1 for a proof). In the lectures we argued that $\tilde{L}$ is a symmetric operator and we know that since it is a generator of a strongly continuous semigroup, it is closed. Also by the Hille Yosida theorem, $(\lambda - \tilde{L})$ is invertible for $\lambda > 0$ and so $(0, \infty) \in \rho(\tilde{L})$, where $\rho(\tilde{L})$ is the resolvent set of $\tilde{L}$.

To make our final conclusion, we need the following lemma about the spectrum of symmetric operators:

Lemma: The spectrum of a closed symmetric operator is either the closure of the upper half plane, the closure of the lower half plane, the whole plane or a subset of the real numbers.

Proof: Consider any point $\lambda \in \rho(\tilde{L})$ (ρ is the resolvent set - $\tilde{L} - \lambda$ is invertible and its inverse, denoted by R_λ, is a bounded linear map). If we now pick some other point $\mu \in \mathbb{C}$ so that $|\mu - \lambda| < ||R_\lambda||^{-1}$, then the series $S = \sum_{n=0}^{\infty} (\lambda - \mu)^n R_\lambda^n$ converges and one can check that $R^\lambda S$ is the inverse of $\tilde{L} - \mu$ so also μ is in the resolvent. This implies that for each $\lambda \in \rho(\tilde{L})$, the disc $B(\lambda, ||R_\lambda||^{-1}) \subset \rho(\lambda)$.

Let us write now $\lambda = \alpha + i\beta$. Then

$$\begin{aligned} ||(\tilde{L} - \lambda)x||^2 &= ((\tilde{L} - \alpha - i\beta)x, (\tilde{L} - \alpha - i\beta)x) \\ &= ((\tilde{L} - \alpha)x, (\tilde{L} - \alpha)x) + \beta^2(x, x) + i\beta(x, (\tilde{L} - \alpha)x) - i\beta((\tilde{L} - \alpha)x, x) \\ &= ||(\tilde{L} - \alpha)x||^2 + \beta^2||x^2||. \end{aligned}$$

So $||(\tilde{L} - \lambda)x||^2 \geq \mathrm{Im}(\lambda)^2||x||^2$. On the other hand, $||(\tilde{L} - \lambda)x||^2 \leq ||R_\lambda||^{-1}||x||^2$ so we see that $||R_\lambda||^{-1} \geq \mathrm{Im}(\lambda)$ so we conclude that $B(\lambda, \mathrm{Im}(\lambda)) \subset \rho(\tilde{L})$. This implies that if a point $\lambda \in \rho(\tilde{L})$ is in the (open) lower half plane, then using such discs, we can actually cover the entire lower half plane. This type of reasoning in the possible cases gives the claimed result.

Since in our case, $(0, \infty) \subset \rho(\tilde{L})$, only $\sigma(\tilde{L}) \subset (-\infty, 0]$ is possible. This implies that $\tilde{L} \pm i$ is invertible and its range is the entire Hilbert space so it must be self adjoint.

2. Consider the setup of the previous problem. Recall that in the lectures we showed that

$$-(f, \tilde{L}g) = \frac{1}{2} \sum_{x,y} \int c(x, y, \eta) \overline{(f(\eta^{x,y}) - f(\eta))} (g(\eta^{x,y}) - g(\eta)) \mu(d\eta)$$

for cylinder functions f, g. Show that this holds for $f, g \in \mathcal{D}(\tilde{L})$ and that the series converges absolutely.

Solution: Consider first the case $-(f, \tilde{L}f)$. Pick now some cylinder functions f_n so that $f_n \to f$ and $Lf_n \to \tilde{L}f$ in L^2 for $f \in \mathcal{D}(\tilde{L})$. Then by Using Fatou's Lemma,

$$\begin{aligned}
-(f, \tilde{L}f) &= -\lim_n (f_n, Lf_n) \\
&= \lim_n \sum_{x,y} \int c(x,y,\eta)|f_n(\eta^{x,y}) - f_n(\eta)|^2 \mu(d\eta) \\
&\geq \sum_{x,y} \int c(x,y,\eta)|f(\eta^{x,y}) - f(\eta)|^2 \mu(d\eta).
\end{aligned}$$

Thus the sum is finite. Note that a Cauchy-Schwarz argument (with respect to the measure $c\mu$ on $\mathbb{Z}^d \times \mathbb{Z}^d \times X$) implies that the series $\sum_{x,y} \int c(x,y,\eta)\overline{(f(\eta^{x,y}) - f(\eta))}(g(\eta^{xy}) - g(\eta))\mu(d\eta)$ converges absolutely. Thus we can use the arguments used in the lectures to obtain the desired result.

3. In addition to the setup of the previous exercises, assume that we are considering the Bernoulli case for the measure μ. Then show that if $g(\eta^{x,y}) = g(\eta)$ for all x, y for μ a.e. η, then g is μ a.e. a constant.

Solution: We can use the Hewitt-Savage 0-1 law for this. To formulate the theorem, we require the concept of permutation invariant sets. Consider a set S and the space of sequences of elements of S, S^∞. Consider a bijection $p : \mathbb{N} \to \mathbb{N}$ so that $p_n = n$ for all but finitely many n. This induces a mapping on S^∞: $T_p((s_n)) = (s_{p_1}, s_{p_2}, ...)$. A set I is symmetric under finite permutations if

$$T_p^{-1} I = \{(s_n) \in S^\infty : T_p(s) \in I\} = I$$

for all finite permutations T_p. One can then check that if $(S, \mathcal{S})$ is a measurable space and $\mathcal{S}^\infty$ is the product σ-algebra on S^∞, then the product measurable symmetric sets form a sub-σ-algebra $\mathcal{I} \subset \mathcal{S}^\infty$. The Hewitt-Savage 0-1 law is a statement about this σ-algebra.

Theorem: Let $X = (X_n)_{n=1}^\infty$ be a sequence of i.i.d random elements in some measurable space $(S, \mathcal{S})$. Then the σ-algebra $X^{-1}\mathcal{I}$ is trivial with respect to the law of X.

For a proof, see Kallenberg. The way we use this, is we consider $X = (\eta(x))_{x \in \mathbb{Z}^d}$. In the Bernoulli-case, $\eta(x)$ are i.i.d. so Hewitt-Savage applies. Consider the Bernoulli-completion of the σ-algebra $X^{-1}\mathcal{I}$ (i.e. we consider the σ-algebra generated by $X^{-1}\mathcal{I}$ and the sets of Bernoulli measure zero). This is still trivial since we are only modifying σ-algebra with sets of measure zero. Now by our assumption (and a small inductive argument if you wish), the event $\{g(\eta) \in A\}$ is in the completed σ-algebra for any measurable set $A \subset \mathbb{R}$. Thus for any set $A \subset \mathbb{R}$, the probability $P(g(\eta) \in A)$ is zero or one. Then we see for example that there is an interval $[n, n+1]$ so that $g(\eta) \in [n, n+1]$ almost surely. We can then split this interval into two parts and see that almost surely $g(\eta)$ is in only one of these. Continuing splitting the intervals into two, we conclude that there is a constant so that $g(\eta)$ equals this constant almost surely.

4. Consider a strongly continuous contraction semigroup $S(t)$ with generator L (whose domain is $\mathcal{D}(L)$). Show that for $f \in \mathcal{D}(L^{n+1})$ (e.g. $f \in \mathcal{D}(L^2)$ if $f \in \mathcal{D}(L)$ and $\lim_{t\to 0} t^{-1}(S(t)Lf - Lf)$ exists). Then show that

$$S(t)f = \sum_{k=0}^{n} \frac{t^k}{k!} L^k f + \frac{1}{n!} \int_0^t (t-s)^n S(s) L^{n+1} f ds.$$

Solution: This is simply and induction and integrating by parts. Integrating by parts is justified by our assumption on f and the fact that the fundamental theorem of calculus holds for Banach space integrals. This is proved in Seppäläinen's notes (Lemma 3.4.).