# STOCHASTIC PARTICLE SYSTEMS: EXERCISE 5 - SOLUTIONS

**1.** During the lectures, we showed that given a strongly continuous contraction semigroup $S(t)$ on a Banach space $B$ and the semigroup's generator $L$, then $B_\lambda$ defined on $B$

$$B_\lambda f = \int_0^\infty e^{-\lambda t} S(t) f dt$$

is the right inverse of the operator $\lambda - L$, i.e., that $(\lambda - L)B_\lambda f = f$ for all $f \in B$. Show that it is also the left inverse (so in fact $B_\lambda = (\lambda - L)^{-1}$).

*Solution:* Let $f \in \mathcal{D}(L)$. We must show that

$$\int_0^\infty e^{-\lambda t} S(t)(\lambda - L) f dt = f.$$

To do this in detail, we need some results from Seppäläinen's notes. First of all, if the function $u : [a, b] \to B$ is continuous, the integral (which we understand as a strong limit of Riemann sums)

$$\int_a^b u(t) dt$$

exists (that is continuous functions are Riemann integrable). We also need that if $K : \mathcal{D}(K) \to B$, where $\mathcal{D}(K) \subset B$, is a closed linear operator, $u : [a, b] \to \mathcal{D}(K)$ is continuous and $Ku$ is continuous, then $\int_a^b u(t) dt \in \mathcal{D}(K)$ and

$$K \int_a^b u(t) dt = \int_a^b K u(t) dt.$$

The proof of these is given in Lemma 3.4 of Seppäläinen's notes. We are interested in the case where $u(t) = S(t)f$ and $K = L$. Using completeness and the fact that $S(t)$ is a contraction, one can make sense of $\int_0^\infty e^{-\lambda t} S(t) f dt$ and also extend the results above to the case of an infinite interval.

Using these results and the fact that $S(t)L = LS(t)$, we have

$$\begin{aligned} B_\lambda(\lambda - L) f &= \int_0^\infty e^{-\lambda t} S(t)(\lambda - L) f dt \\ &= \lambda B_\lambda f - \int_0^\infty L e^{-\lambda t} S(t) f dt \\ &= \lambda B_\lambda f - L \int_0^\infty e^{-\lambda t} S(t) f dt \\ &= (\lambda - L) B_\lambda f \\ &= f. \end{aligned}$$

In the last step we used the result we proved in the lectures.

**2.** Consider the Banach space $C(X)$, where $X$ is a compact metric space with its measurable structure given by the Borel sets.

a) Suppose $L$ is a Markov pregenerator with domain $\mathcal{D}(L) \subset C(X)$. Show that $L$ has a closure $\bar{L}$ and this is a Markov pregenerator as well.

b) Suppose that $L$ is a closed Markov pregenerator with domain $\mathcal{D}(L) \subset C(X)$. Show that for $\lambda > 0$, the range of $\lambda - L$ is a closed subset of $C(X)$.

*Solution:* a) Recall that a closed linear operator on $C(X)$ is one who's graph $\{(f, Lf) : f \in \mathcal{D}(L)\}$ is a closed subset of $C(X) \times C(X)$. The closure of $L$ is the smallest closed extension of $L$ (smallest meaning that the domain is the smallest with respect to inclusion of sets). Pick now $f_n \in \mathcal{D}(L)$ so that $f_n \to 0$ and $Lf_n \to h$ for some $h \in C(X)$. Moreover, let $g \in \mathcal{D}(L)$ be arbitrary. Recalling that we proved in the lectures that for $f \in \mathcal{D}(L)$, one has $||(\lambda - L)f|| \geq \lambda ||f||$ for $\lambda \geq 0$, we see that for $\lambda \geq 0$

$$||(\lambda - L)(\lambda f_n + g)|| \geq \lambda ||\lambda f_n + g||.$$

Taking $n \to \infty$, we have

$$||\lambda g - \lambda h - Lg|| \geq \lambda ||g||.$$

Dividing by $\lambda$ and taking $\lambda \to \infty$, we see that

$$||g - h|| \geq ||g||.$$

This must hold for any $g \in \mathcal{D}(L)$. Recall that $\mathcal{D}(L)$ is dense in $C(X)$ by the definition of a Markov pregenerator so in fact $||v - h|| \geq ||v||$ for any $v \in C(X)$. Choosing $v = h$ shows that $h = 0$. This implies that if $f \in C(X)$ so that for some $f_n \in \mathcal{D}(L)$ $f_n \to f$ and $Lf_n$ converges, then $\bar{L}f = \lim_n Lf_n$ is well defined (i.e. independent of the sequence $f_n$ we chose). Also the graph of $\bar{L}$ is then $\overline{\{(f, Lf) | f \in \mathcal{D}(L)\}}$ so $\bar{L}$ is closed and its domain is minimal, i.e., it is the closure of $L$.

To check that it is a Markov pregenerator, it is enough to check that if $f \in \mathcal{D}(\bar{L})$, $\lambda > 0$ and $g = (\lambda - \bar{L})f$, then $\lambda \min_{\eta \in X} f(\eta) \geq \min_{\eta \in X} g(\eta)$ (the other properties of a pregenerator follow directly from $L$). By our construction of $\bar{L}$, we have a sequence $f_n \in \mathcal{D}(L)$ so that $f_n \to f$ and $Lf_n \to \bar{L}f$. Let $g_n = (\lambda - L)f_n$. Since $L$ is a Markov pregenerator, $\lambda \min_{\eta \in X} f_n(\eta) \geq \min_{\eta \in X} g_n(\eta)$. Since $g_n \to g$ and we are dealing with continous functions on a compact space, can take $n \to \infty$ in this relation and conclude that $\bar{L}$ is a Markov pregenerator.

b) Pick $g_n$ from the range of $\lambda - L$ so that $g_n \to g$. Let $f_n \in \mathcal{D}(L)$ so that $(\lambda - L)f_n = g_n$. Then $\lambda(f_n - f_m) - L(f_n - f_m) = g_n - g_m$ and using $||(\lambda - L)f|| \geq \lambda ||f||$ gives $\lambda ||f_n - f_m|| \leq ||g_n - g_m||$. Since $g_n$ is Cauchy, this implies that $f_n$ is Cauchy and a limit $f = \lim_n f_n$ exists. Now $Lf_n = \lambda f_n - g_n \to \lambda f - g$ and since $L$ is closed, $g = (\lambda - L)f$ and $g$ is in the range of $\lambda - L$ so the range is a closed set.

**3.** Let $\rho, \rho' \in (0, 1)$ and $\rho \neq \rho'$. Show that in finite volume the Bernoulli measures $\nu_\rho$ and $\nu_{\rho'}$ are absolutely continuous with respect to each other, but in infinite volume they are not.

*Solution:* In finite volume the only set that has zero Bernoulli measure for any parameter $\rho$ is the empty set so absolute continuity is clear in the finite volume case. For the infinite volume case, consider Bernoulli measures on $\mathbb{Z}$ (extension to higher dimensions will follow from similar constructions).

Let $(I_n)$ be a sequence of finite intervals of $\mathbb{Z}$ so that $x < y$ for $x \in I_n$ and $y \in I_{n+1}$ and $\cup_n I_n = \{0, 1, 2, ...\}$. Let $i_n = |I_n|$ be the number of points in $I_n$. Consider now the event $A$ that for each interval $I_n$ there is a site in $I_n$ so that there is no particle present there. The Bernoulli-$\rho$ probability of this event is then

$$\nu_\rho(A) = \prod_{n=1}^{\infty}(1 - \rho^{i_n}).$$

Thus

$$\log \nu_\rho(A) = \sum_{n=1}^{\infty} \log(1 - \rho^{i_n}).$$

Let us assume that $\rho < \rho'$. Our goal is to pick the $i_n$ so that for one of the parameters $\rho$ or $\rho'$ the above series diverges to $-\infty$ (so that the probability is zero) while for the other it stays finite. First of all, we note that

$$\sum_{n=1}^{\infty} \log(1 - (\rho')^{i_n}) \leq -\sum_{n=1}^{\infty}(\rho')^{i_n}.$$

So if we pick $i_n = -[\frac{\log n}{\log \rho'}]$, the series is comparable with the harmonic series and it diverges to $-\infty$.

On the other hand, we can find an $\alpha > 0$ so that $\log(1 - x) \geq -\alpha x$ for $x \in [0, \rho]$ (e.g. $\alpha = \frac{1}{1-\rho}$) and

$$\sum_{n=1}^{\infty} \log(1 - \rho^{i_n}) \geq -\alpha \sum_{n=1}^{\infty} \rho^{i_n}.$$

For the $i_n$ above, the series is comparable with the series

$$-\alpha \sum_{n=1}^{\infty} n^{-\frac{\log \rho}{\log \rho'}}.$$

Since $0 < \rho < \rho' < 1$, $\frac{\log \rho}{\log \rho'} > 1$ and this series converges to a finite value. $\nu_{\rho'}(A) = 0$ but $\nu_\rho(A) > 0$. To find an event $A'$ so that the roles of $\rho$ and $\rho'$ are reversed, consider just the event where on each interval there is a site where there is a particle present.

**4.** a) Show that the following are Markov pregenerators.

i) $L = T - 1$, where $T$ maps non-negative functions of $C(X)$ into non-negative functions of $C(X)$ ($X$ as in problem 2), is defined on all of $C(X)$ and satisfies $T1 = 1$.

ii) Let $X = [0, 1]$ and $Lf = \frac{1}{2}f''$ with $\mathcal{D}(L) = \{f \in C(X) : f'' \in C(X), f'(0) = f'(1) = 0\}$.

iii) $X = [0, 1]$ and $Lf = \frac{1}{2}f''$ with $\mathcal{D}(L) = \{f \in C(X) : f'' \in C(X), f''(0) = f''(1) = 0\}$.

b) Show that all of these Markov pregenerators are actually Markov generators. What are the corresponding Markov processes?

*Solution:* a) To check that the domains in ii) and iii) are dense, we shall need the following version of the Stone-Weierstrass theorem.

**Theorem:** Let $X$ be a compact Hausdorff space and $B$ be a closed subalgebra of $C(X)$. If $1 \in B$ and $B$ separates points (i.e. for all $x, y \in X$ so that $x \neq y$ we can find a $f \in B$ such that $f(x) \neq f(y)$) then $B = C(X)$.

For a proof, see e.g. Reed and Simon: Methods of Mathematical physics vol 1.

i) Recall that we had three conditions to check for an operator to be a Markov pregenerator. First of all since $\mathcal{D}(L) = C(X)$, we see that $\mathcal{D}(L)$ is dense and contains the constant function 1. Moreover, by definition $L1 = 0$. We recall that a sufficient condition for the last condition was the minimum principle: if $f \in \mathcal{D}(L)$ and $f(\eta) = \min_{\xi \in X} f(\xi)$, then $Lf(\eta) \geq 0$. To check this, note that if $\eta$ is as above, $g = f - f(\eta)$ is a non-negative continuous function. Also $Tg = Tf - f(\eta)T1 = Tf - f(\eta)$. Since this must be a non-negative function, $Tf(\eta) \geq f(\eta)$. Thus $Lf(\eta) \geq 0$ and $L$ is a Markov pregenerator.

ii) Clearly $1 \in \mathcal{D}(L)$ and $L1 = 0$. $\mathcal{D}(L)$ is dense by Stone-Weierstrass. Now if $f(\eta) = \min_{\xi \in [0,1]} f(\xi)$, then $f'(\eta) = 0$ (since $f'$ vanished at the boundary according to our definition). Thus the leading order behavior is determined by $f''(\eta)$. If $f''(\eta) < 0$, then $\eta$ could not be a minimum so $Lf(\eta) \geq 0$.

iii) Again $1 \in \mathcal{D}(L)$ and $L1 = 0$ are clear. To apply Stone-Weierstrass, we note that now $\mathcal{D}(L)$ is not an algebra, but it contains the algebra $\{f \in C(X) : f'' \in C(X), f'(0) = f'(1) = f''(0) = f''(1)\}$ which is dense by Stone-Weierstrass. Let $\eta$ be a minumum point of $f \in \mathcal{D}(L)$. Then either $\eta$ is a boundary point (and $Lf(\eta) = 0$) or $f'(\eta) = 0$ and the behavior is determined by $f''(\eta)$ so we conclude as in ii) that $Lf(\eta) \geq 0$ in this case as well.

b) We need to check that the operators are closed and that the range of $\lambda - L$ is $C(X)$ for large enough $\lambda$.

i) $L$ is everywhere defined, i.e., bounded (continuous) so it is automatically closed. That the range for $\lambda - L$ is $C(X)$ for large enough $\lambda$ follows simply from the von Neumann series: pick $g \in C(X)$, then for

$$f = \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{1}{\lambda^k} L^k g$$

$(\lambda - L)f = g$. The series converges for large enough $\lambda$ since $L$ is bounded.

$T$ is a bounded linear mapping on $C(X)$. Thus for each $x$, the mapping $T_x : C(X) \to \mathbb{R}$, $T_x f = (Tf)(x)$ is a positive bounded linear functional on $C(X)$ so by the Riesz representation theorem, $T_x f = \int_X f(y)\mu(x, dy)$ for some kernel $\mu$. Thus $Tf = \int_X f(y)\mu(\cdot, dy)$.Since $T1 = 1$, $\mu$ is a probability kernel (i.e. $\mu(x, X) = 1$ for all $x$).

Consider now the pure jump Markov process $X_t$ for which the jump times are exponential with rate 1 and for which the jump from $x$ is distributed according to $\mu(x, \cdot)$. Then

$$
\begin{aligned}
E^x(f(X_t)) &= \sum_{n=0}^{\infty} e^{-t}\frac{t^n}{n!}\int_{X^n}\mu(x,dy_1)\mu(y_1,dy_2)\cdot\ldots\cdot\mu(y_{n-1},dy_n)f(y_n)\\
&= e^{-t}e^{tT}f(x)\\
&= e^{t(T-1)}f(x)\\
&= e^{tL}f(x).
\end{aligned}
$$

So we conclude that $L$ is the generator of a pure jump Markov process who's jump times are exponential with rate 1 and jumps are distributed according to $\mu$.

ii),iii) That the operators are closed, follows from basic results concerning uniform convergence and differentiation (if $f_n$ converges uniformly to $f$ and $f_n'$ converges uniformly to $g$, then $f' = g$). To show that the range of $\lambda - L$ is $C(X)$ one has to show that the boundary value problem $\lambda f - \frac{1}{2}f'' = g$, $f'(0) = f'(1) = 0$ (or $f''(0) = f''(1) = 0$) has a solution for each $g \in C(X)$. This follows from basic theory of differential equations (variation of constants).

Consider Brownian motion which is reflected at the origin. We can write this process as $|B_t|$ where $B_t$ is a standard Brownian motion. The transition function for the the reflected process is

$$
p_t(x,y) = \frac{1}{\sqrt{2\pi t}}\left(e^{-\frac{(x-y)^2}{2t}} + e^{-\frac{(x+y)^2}{2t}}\right)
$$

where $x, y \geq 0$ (we start the process at $x \geq 0$). Then for any $C^2[0,\infty)$

$$
\frac{E^0(f(|B_t|)) - f(0)}{t} = \frac{1}{t}\left(\int_0^{\infty}\left(\frac{1}{\sqrt{2\pi}}2e^{-\frac{y^2}{2}}(f(0) + \sqrt{t}yf'(0) + \mathcal{O}(t))\right)dy - f(0)\right).
$$

This can only have a limit if $f'(0) = 0$. So it stands to reason that if we consider Brownian motion reflected at 0 and 1, then we are in the case with the Neumann boundary conditions.

Consider now the case where the boundary conditions are $f''(0) = f''(1) = 0$. As the generator is still the Laplacian, we expect that the process will again be Brownian motion with some special properties near the boundary. Let us start the process at 0 and consider the Markov evolution $S(t)f(0) = \mathbb{E}^0(f(X_t))$ for some function $f$ in the domain of the generator. From the properties of Markov generators and the definition of $L$, we see that

$$
\frac{d}{dt}S(t)f(0) = LS(t)f(0) = S(t)(Lf)(0) = 0.
$$

Thus $S(t)f(0) = f(0)$. So if we start the process at 0, it stays at zero for all times. Using the same argument, this happens if we start the process at 1. So we conclude that our process is Brownian motion which is absorbed at 0 and 1, i.e. if the process is started at $x \in (0,1)$, it behaves like a normal Brownian motion untill it hits either the point 0 or 1 and when it hits one of the boundary points, it remains at this point for all of eternity.