STOCHASTIC PARTICLE SYSTEMS - EXERCISE 4: SOLUTIONS

This week all of the exercises are from Seppäläinen's notes.

1. Seppäläinen Exercise 2.2 - page 36 (The event that jumps don't occur simultanously, for each x there are only finitely many jumps to and from x in any finite time interval and that all connected components are finite in suitable time intervals is measurable and has probability one).

Solution: Let $\{\mathcal{T}_{(x,y)}\}$ be the collection of independent Poisson processes (which we view as random points on $(0,\infty)$) used to construct the exclusion process ($\mathcal{T}_{(x,y)}$ has rate $p(x,y)$). We then define $\mathcal{T}'_x = \cup_y(\mathcal{T}_{(x,y)} \cup \mathcal{T}_{(y,x)})$ to be the set of times when a jump to or from x can occur. As mentioned in Seppäläinen's notes, $\mathcal{T}'_x$ is a Poisson process with rate 2. We then write $\tilde{\Omega}$ for the set of realizations of the collection of Poisson processes so that for every $T \in (0,\infty)$, $\mathcal{T}'_x$ has only finitely many points in $[0,T]$ and for $(x,y) \neq (x',y')$, $\mathcal{T}_{(x,y)}$ and $\mathcal{T}_{(x',y')}$ are disjoint, i.e. there is no common jump time.

Since we have a countable collection of Poisson processes, the measurability of the event that $\mathcal{T}'_x$ has only finitely many points in $[0,T]$ is clear. Again by countability the event that for $(x,y) \neq (x',y')$, $\mathcal{T}_{(x,y)}$ and $\mathcal{T}_{(x',y')}$ are disjoint is measurable. So $\tilde{\Omega}$ is a measurable set. To show that it has probability one, we note that $\mathcal{T}'_x$ has only finitely many points in $[0,T]$ a.s. since $\mathcal{T}'_x$ is a Poisson process as well. That $\mathcal{T}_{(x,y)}$ and $\mathcal{T}_{(x',y')}$ are a.s. disjoint for $(x,y) \neq (x',y')$ follows from the fact that we are dealing with continunous distributions. So $\tilde{\Omega}$ has probability one.

Let Ω_1 be the event that the random graph $\mathcal{G}_{0,t_0}$ (containing of all edges that represent a jump before time t_0) has only finite connected components. We consider its complement. Recall that by translation invariance, this is equivalent to the origin being in an infinite connected component. Consider then the event A_m that there is a self avoiding path of length m starting from the origin that stays in the connected component of the origing. This is an event one can construct with countable operations from the Poisson realizations. Thus $\limsup A_m$ (i.e. the event that there are arbitrarily long paths) is also measurable and by the Borel-Cantelli argument in Seppäläinen's notes, it has measure zero. But this means that Ω_1 is measurable and has probability one. We then repeat the argument for each $G_{(k-1)t_0,kt_0}$ (and have a set Ω_k). So taking $\tilde{\Omega} \cap \cap_{k=1}^{\infty}\Omega_k$ shows that we have a measurable set with probability one so that there are no infinite connected components and the jumps behave nicely.

2. Seppäläinen Exercise 2.3. - page 36 (The generator of the exclusion process is not defined on all continuous functions and it is not continuous on cylinder functions).

Solution: First we wish to construct a continuous function $f : \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}$ so that Lf is not defined. So it is enough to find a point $\eta \in \{0,1\}^{\mathbb{Z}^d}$ so that $Lf(\eta) = -\infty$. Let us consider a very simple case: $d=1$ and $p(x,y) = \delta_{y,x+1}$ i.e. the totally asymetric case in one dimension. Let us now define the sets $A_n = \{-2n, -2(n-1), ... -2, 0, 2, ..., 2n\}$ and the function $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$ by

$$f(\eta) = \sum_{n=1}^{\infty} \frac{1}{n^2} \eta_{A_n},$$

where $\eta_{A_n} = \prod_{x \in A_n} \eta(x)$. This is a continuous function. To see this, consider two configurations η and η'. Recall that the metric on $\{0,1\}^{\mathbb{Z}}$ was defined so that η and η' are close to each other if they agree in some large set around the origin. This means that $\eta_{A_n} = \eta'_{A_n}$ for all $n \leq m$ where m is some large number. Then $|f(\eta) - f(\eta')| \leq \sum_{n=m}^{\infty} \frac{1}{n^2}$ which is small when m is large. Thus f is continuous.

Consider now the configuration η where $\eta(x) = 1$ when x is even and $\eta(x) = 0$ when x is odd. In the totally asymmetric case, we have for this configuration

$$\begin{aligned} Lf(\eta) &= \sum_{x,y \in \mathbb{Z}} \delta_{y,x+1} \eta(x)(1 - \eta(y))(f(\eta^{x,y}) - f(\eta)) \\ &= \sum_{x \in 2\mathbb{Z}} (f(\eta^{x,x+1}) - f(\eta)) \\ &= \sum_{x \in 2\mathbb{Z}} \sum_{n=1}^{\infty} \frac{1}{n^2} (\eta_{A_n}^{x,x+1} - \eta_{A_n}). \end{aligned}$$

We now note that if $x \notin A_n$, $\eta_{A_n}^{x,x+1} = \eta_{A_n}$. If $x \in A_n$, then for our specific configuration η, $\eta_{A_n}^{x,x+1} = 0$ (since $x+1$ odd so for our configuration $\eta^{x,x+1}(x) = \eta(x+1) = 0$ so the product $\eta_{A_n}^{x,x+1}$ is zero for $x \in A_n$). On the other hand $\eta_{A_n} = 1$ for the configuration we are considering. Thus for $x \in A_n$, $\eta_{A_n}^{x,x+1} - \eta_{A_n} = -1$. So we have

$$Lf(\eta) = -\sum_{x \in 2\mathbb{Z}} \sum_{n=1}^{\infty} \frac{1}{n^2} \mathbf{1}(x \in A_n).$$

We are summing over non-negative terms so we can switch the order of the sums:

$$\begin{aligned} Lf(\eta) &= -\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{x \in 2\mathbb{Z}} \mathbf{1}(x \in A_n) \\ &= -\sum_{n=1}^{\infty} \frac{1}{n^2} (2n+1) \\ &= -\infty. \end{aligned}$$

Thus Lf is not defined for all continuous functions f. To show that L is not continuous even among cylinder functions, consider the cylinder functions $f_m(\eta) = \sum_{n=1}^{m} \frac{1}{n^2} \eta_{A_n}$ now $||f_m|| \leq \frac{\pi^2}{6}$ but arguing as above, we see that

$$||Lf_m|| \geq \sum_{n=1}^{m} \frac{2n+1}{n^2}$$

which is unbounded.

3. Seppäläinen Exercise 2.5. - page 36 (Showing directly without semigroup theory that $M_t = f(\eta_t) - \int_0^t Lf(\eta_s)ds$ is a martingale).

Remark: In the solution, most statements and equation hold only almost surely, but for brevity, we refrain from writing this every time.

Solution: Let f be a cylinder function on $X = \{0,1\}^{\mathbb{Z}^d}$. What we want to show is that for $t > s$ $E(M_t - M_s|\mathcal{F}_s) = 0$, where $\mathcal{F}_s$ is the filtration of the exclusion process η_t. Let us partition the interval $[s,t]$ into m subintervals $[s_i, s_{i+1}]$, where $s_{i+1} - s_i = \delta = \frac{t-s}{m}$ (so $s_1 = s$, $s_{m+1} = t$). We then have

$$\begin{aligned} E(M_t - M_s|\mathcal{F}_s) &= E\left(f(\eta_t) - f(\eta_s) - \int_s^t Lf(\eta_u)du \middle| \mathcal{F}_s \right) \\ &= E\left(\sum_{i=1}^m \left(f(\eta_{s_{i+1}}) - f(\eta_{s_i}) - \int_{s_i}^{s_{i+1}} Lf(\eta_u)du \right) \middle| \mathcal{F}_s \right). \end{aligned}$$

Using the 'tower property of conditional expectation' (i.e. that for $\mathcal{F} \subset \mathcal{G}$, $E(E(X|\mathcal{G})|\mathcal{F}) = E(X|\mathcal{F})$), we note that $E(f(\eta_{s_{i+1}})|\mathcal{F}_s) = E(E(f(\eta_{s_{i+1}})|\mathcal{F}_{s_i})|\mathcal{F}_s)$. So it is enough to show that

$$E\left(\sum_{i=1}^m \left(E(f(\eta_{s_{i+1}})|\mathcal{F}_{s_i}) - f(\eta_{s_i}) - \int_{s_i}^{s_{i+1}} Lf(\eta_u)du \right) \middle| \mathcal{F}_s \right) = 0.$$

Now using the Markov property of the exclusion process and rearranging the terms slightly shows that

$$\begin{aligned} &E(f(\eta_{s_{i+1}})|\mathcal{F}_{s_i}) - f(\eta_{s_i}) - \int_{s_i}^{s_{i+1}} Lf(\eta_u)du \\ &= E^{\eta_{s_i}}(f(\eta_\delta)) - f(\eta_{s_i}) - \delta Lf(\eta_{s_i}) - \delta(Lf(\eta_{s_{i+1}}) - Lf(\eta_{s_i})) + \int_{s_i}^{s_{i+1}} (Lf(\eta_{s_{i+1}}) - Lf(\eta_u))du. \end{aligned}$$

For the first three terms, we note that (Seppäläinen (2.14))

$$\begin{aligned} |E^{\eta_{s_i}}(f(\eta_\delta)) - f(\eta_{s_i}) - \delta Lf(\eta_{s_i})| &= |S(\delta)f(\eta_{s_i}) - f(\eta_{s_i}) - \delta Lf(\eta_{s_i})| \\ &\leq \sup_{\eta \in X} |S(\delta)f(\eta) - f(\eta) - \delta Lf(\eta)| \\ &\leq C(f)\delta^2, \end{aligned}$$

where $C(f)$ is some constant depending on f. Thus

$$\sum_{i=1}^m |E^{\eta_{s_i}}(f(\eta_\delta)) - f(\eta_{s_i}) - \delta Lf(\eta_{s_i})| \leq C(f)(t-s)\delta.$$

For the next term, we note that

$$\sum_{i=1}^m \delta(Lf(\eta_{s_{i+1}}) - Lf(\eta_{s_i})) = \delta(Lf(\eta_t) - Lf(\eta_s)).$$

For a cylinder function f, $Lf \in C(X)$ and there exists a constant $B(f)$ so that $||Lf||_\infty \le B(f) < \infty$ (Seppäläinen (2.13) or a simple argument from the formula for the generator and the definition of a cylinder function). Thus

$$\left| \sum_{i=1}^{m} \delta(Lf(\eta_{s_{i+1}}) - Lf(\eta_{s_i})) \right| \le 2B(f)\delta.$$

So we see that

$$|E(M_t - M_s|\mathcal{F}_s)| \le C(f)(t-s)\delta + 2B(f)\delta + E\left(\left| \sum_{i=1}^{m} Lf(\eta_{s_{i+1}})\delta - \int_s^t Lf(\eta_u)du \right| \middle| \mathcal{F}_s \right).$$

Since η is RCLL and $Lf \in C(X)$, $(Lf) \circ \eta$ is RCLL so $(Lf) \circ \eta$ is Riemann integrable. Thus

$$\sum_{i=1}^{m} Lf(\eta_{s_{i+1}})\delta - \int_s^t Lf(\eta_u)du \to 0$$

as $m \to \infty$. Again since $||Lf||_\infty < \infty$, dominated convergence implies that

$$E\left(\left| \sum_{i=1}^{m} Lf(\eta_{s_{i+1}})\delta - \int_s^t Lf(\eta_u)du \right| \middle| \mathcal{F}_s \right) \to 0$$

as $m \to \infty$. Since $\delta \to 0$ as $m \to \infty$ and the left side of the equation is independent of m, we see that $E(M_t - M_s|\mathcal{F}_s) = 0$.

4. Seppäläinen Exercise 3.5. - page 50 (Semigroup formalism in a couple of deterministic cases).

Solution: a) For $(S(t)f)(x) = f(x + at)$, there are several 'natural' Banach spaces on which $S(t)$ can act and be strongly continuous. For example (see Rudin: Real and Complex analysis - Theorem 9.5) for $p \in [1, \infty)$ and $f \in L^p(\mathbb{R})$, the mapping $t \mapsto f_t$, where $f_t(x) = f(x - t)$ is a uniformly continuous mapping of $\mathbb{R}$ into $L^p(\mathbb{R})$. In particular, for each $\epsilon > 0$, there is a $\delta > 0$ so that for $0 \le t < \delta$, $||S(t)f - f||_p = ||f_{-at} - f_0||_p < \epsilon$ so S is strongly continuous on $L^p(\mathbb{R})$ for $p \in [1, \infty)$.

During the course, we have mainly been interested in Feller processes. According to our (Seppäläinen's) definition, Feller processes act on bounded continuous mappings. So another place to look for suitable Banach spaces is subspaces of $C_b(\mathbb{R})$ (the bounded continuous functions on $\mathbb{R}$) with the sup-norm. In this framework, the strong continuity of S becomes a question of uniform continuity. So we want a Banach space of bounded continuous functions which are uniformly continuous.

This in fact requires a proper subspace of $C_b(\mathbb{R})$. To see this, note that uniform continuity implies that for any sequences $(x_n), (y_n)$ so that $|x_n - y_n| \to 0$, one has $|f(x_n) - f(y_n)| \to 0$. One then considers the function $x \mapsto \sin x^2$ and the sequences $x_n = \sqrt{(n + \frac{1}{2})\pi}$ and $y_n = \sqrt{n\pi}$. Then $|x_n - y_n| \to 0$ but $|\sin x_n^2 - \sin y_n^2| = 1$. So not all bounded continuous functions are uniformly continuous and $S(t)$ is not strongly continuous on $C_b(\mathbb{R})$. One natural subspace of $C_b(\mathbb{R})$ for which all functions in it are uniformly continuous is $C_0(\mathbb{R})$ - the space of continuous functions vanishing at infinity (i.e. $\lim_{x \to \pm\infty} f(x) = 0$). In fact, many books define Feller processes to act on $C_0(\mathbb{R})$

instead of $C_b(\mathbb{R})$. So depending on one's approach, one can consider S as acting on the Banach spaces $C_0(\mathbb{R})$ or $L^p(\mathbb{R})$ with $p \in [1, \infty)$ (or perhaps some closed subspaces of these).

That $Lf = af'$ follows from the definition of the generator. A heuristic argument for this also comes from considering a Taylor series expansion of a function f. Formally one can write a Taylor series expansion as $f(x + a) = e^{a\frac{d}{dx}} f(x)$. So this would suggest that $S(t) = e^{ta\frac{d}{dx}}$ and since one wants to think of the generator L as something that satisfies $S(t) = e^{tL}$, one would then get $L = a\frac{d}{dx}$.

Let us now consider an arbitrary $g \in C_0(\mathbb{R})$ and $f \in \mathcal{D}(L)$ (as $L(\mathcal{D}(L)) \subset C_0(\mathbb{R})$ we note that f is continuously differentiable and a derivative which vanishes at infinity) so that

$$-af' + \lambda f = g.$$

We can of course consider this equation in a moving coordinate system: for any $t \geq 0$ and $x \in \mathbb{R}$

$$-af'(x + at) + \lambda f(x + at) = g(x + at).$$

Now we can consider the derivative acting on f to be with respect to t instead of x and we have

$$-\frac{d}{dt} f(x + at) + \lambda f(x + at) = g(x + at).$$

Multiplying by $e^{-\lambda t}$ and integrating over $[0, \infty)$ (note that everything works nicely since we are dealing with $C_0(\mathbb{R})$ functions) we get (by integrating by parts)

$$\begin{aligned} \int_0^\infty e^{-\lambda t} g(x + at) dt &= -\int_0^\infty e^{-\lambda t} \frac{d}{dt} f(x + at) dt + \lambda \int_0^\infty e^{-\lambda t} f(x + at) dt \\ &= \Big/_{0}^{\infty} -e^{-\lambda t} f(x + at) + \int_0^\infty \frac{d}{dt} \left(e^{-\lambda t}\right) f(x + at) dt + \lambda \int_0^\infty e^{-\lambda t} f(x + at) dt \\ &= f(x). \end{aligned}$$

So we have showed that if $(\lambda - L)f = g$, f is in the domain of L (which is a subset of $C_0(\mathbb{R})$) and $g \in C_0(\mathbb{R})$, then

$$f(x) = \int_0^\infty e^{-\lambda t} (S(t)g)(x) dt.$$

Then using the integration theory in Banach spaces discussed in Seppäläinen, one can argue that this is an instance of a general formula called the resolvent formula:

$$(\lambda - L)^{-1} g = \int_0^\infty e^{-\lambda t} S(t) g dt.$$

Note that in Banach spaces there are several concepts of the integral. We (following Seppäläinen) consider Riemann integration in Banach spaces. That is we consider integrals to be strong limits of Riemann sums. The Bochner integral is the generalization of the Lebesgue integral to Banach spaces. The idea of the Bochner integral is to first define it for simple functions in the natural way and then for measurable functions as a limit of the simple function case. Finally there is the Pettis or weak integral which is defined through duality. A function $f : X \to B$ (X is some measure space and B is a Banach space) is weakly integrable if there is a vector $v \in B$ so that $\langle \phi, v \rangle = \int \langle \phi, f(t) \rangle \mu(dt)$ for all bounded linear functionals ϕ. Then one writes $v = \int f(t)\mu(dt)$.

b) We now consider the simple case where our Banach space is just $\mathbb{R}$ and $S(t)x = e^{-at}x$ for some fixed a. Then $Lx = -ax$ and $\mathcal{D}(L) = \mathbb{R}$. Consider now the equation

$$(\lambda - L)x = y.$$

The solution of this is of course $x = \frac{1}{a+\lambda}y$. On the other hand

$$\int_0^\infty e^{-\lambda t} S(t) x dt = x \int_0^\infty e^{-(\lambda+a)t} dt = \frac{1}{\lambda + a} x.$$

So we have another instance of the resolvent formula.