STOCHASTIC PARTICLE SYSTEMS: EXERCISE 3 - SOLUTIONS

1. Let $(\mathcal{T}_i)_{i=1}^\infty$ be independent Poisson processes with rates $(r_i)_{i=1}^\infty$ satisfying $\sum_{i=1}^\infty r_i < \infty$. Let $\mathcal{T} = \cup_{i=1}^\infty \mathcal{T}_i$. Show that $\mathcal{T}$ is a Poisson process with rate $r = \sum_i r_i$. Moreover, for any $s \in (0, \infty)$, show that first point in $\mathcal{T}$ larger than s comes from the set $\mathcal{T}_i$ with probability $\frac{r_i}{r}$.

Solution: As the laws of random variables are characterized by their Laplace transformations, the laws of random measures μ (such as $\sum_{t\in\mathcal{T}} \delta_t$ for the Poisson process) are characterised by their Laplace functionals $f \mapsto E(\exp(-\int f d\mu))$, where f is a positive measurable function (or if μ is a random measure on a locally compact second countable Hausdorff space it is enough to consider positive compactly supported continuous functions). For a proof of this, see for example the beginning of chapter 12 of Kallenberg's Foundations of Modern Probability.

So to show that $\mathcal{T}$ is a Poisson process, it is enough to consider its Laplace functional. Let us write N_i for the random measure associated to the point set $\mathcal{T}_i$ and $N = \sum_i N_i$, which is the random measure associated with $\mathcal{T}$ (recall that the random measures are just sums of δ-measures).

$$
\begin{aligned}
E(\exp(-\int f dN)) &= \prod_i E(\exp(-\int f dN_i)) \\
&= \prod_i \exp\left(-r_i \int_0^\infty \left(1 - e^{-f(x)}\right) dx\right) \\
&= \exp\left(-r \int_0^\infty \left(1 - e^{-f(x)}\right) dx\right).
\end{aligned}
$$

This is just the Laplace functional of the Poisson process with rate r so we see that indeed $\mathcal{T}$ is a Poisson process with rate r.

Since $\mathcal{T}$ is a Poisson process, there are almost surely only finitely many points of the Process on each interval (s,t). Thus for each $s > 0$, $\tau_s = \min\{T \in \mathcal{T} : T > s\}$ is almost surely well defined and one can check that it is a random variable (i.e. it is measurable). We are interested in the probability $P(\tau_s \in \mathcal{T}_i)$. The event $\{\tau_s \in \mathcal{T}_i\}$ can be written as $\{N_i(s,\tau_s] = 1\}$ or with some redundancy as $\{N_i(s,\tau_s] = 1, N(s,\tau_s] = 1\}$. Thus

$$
P(\tau_s \in \mathcal{T}_i) = P(N_i(s,\tau_s] = 1 | N(s,\tau_s] = 1).
$$

let us now consider for any $\epsilon > 0$, the probability $P(N^i(s, s+\epsilon] = 1 | N(s, s+\epsilon] = 1)$. Noting that $N^i(s, s+\epsilon]$ is Poisson with parameter $r_i\epsilon$ (and similarly for N and N^j for $j \neq i$) and using the independence of N^i and N^j for $j \neq i$,

$$\begin{aligned} P(N^i(s,s+\epsilon]=1|N(s,s+\epsilon]=1) &= \frac{P(N^i(s,s+\epsilon]=1, N^j(s,s+\epsilon]=0, \text{ for } j\neq i)}{P(N(s,s+\epsilon]=1)} \\ &= \frac{e^{-r_i\epsilon}r_i\epsilon\prod_{j\neq i}e^{-r_j\epsilon}}{e^{-r\epsilon}r\epsilon} \\ &= \frac{r_i}{r}. \end{aligned}$$

Then conditioning (or disintegrating) $P(N_i(s,\tau_s]=1|N(s,\tau_s]=1)$ on τ_s gives the desired result.

2. Let $\mathcal{T}=\{T_1,T_2,...\}$ be a Poisson process. Show that given T_{n+1}, the distribution of $(T_1,...,T_n)$ is the distribution of $(S_1,...,S_n)$ where $(S_1,..,S_n)$ is gotten from i.i.d. uniform random variables on $[0,T_{n+1}]$ by putting the random variables into increasing order.

Remark: A related and perhaps a bit more rigorous formulation and solution of the problem would be to show a similar results for variables on $[0,1]$ by defining $U_i=\frac{T_i}{T_{n+1}}$ and $V_i=\frac{S_i}{T_{n+1}}$.

Solution: Let P be the law of $(T_1,...,T_{n+1})$ and Q the law of $(S_1,...,S_n)$. We wish to show that $P(A|T_{n+1})=Q(A)$. One way to do this, is to show that for all continuous functions $f:[0,T_{n+1}]^n\to\mathbb{R}$, $E_P(f(T_1,...,T_n)|T_{n+1})=E_Q(f(S_1,...,S_n))$. By definition,

$$E_Q(f(S_1,...,S_n))=\frac{n!}{T_{n+1}^n}\int_0^{T_{n+1}}ds_n\int_0^{s_n}ds_{n-1}...\int_0^{s_2}ds_1 f(s_1,s_2,...,s_n).$$

Note that the factor $n!$ comes from the fact that we condition n uniform random variables on $[0,T_{n+1}]$ to be in increasing order. By symmetry, the event we condition on has probability $\frac{1}{n!}$.

On the other hand, if λ is the rate of the Poisson process, the conditional density of $(T_1,...,T_n)$ given T_{n+1} is

$$\begin{aligned} \frac{\prod_{i=1}^{n+1}e^{-\lambda(t_i-t_{i-1})}}{\int_0^{T_{n+1}}dt_n\int_0^{t_n}dt_{n-1}...\int_0^{t_2}dt_1\prod_{i=1}^{n+1}e^{-\lambda(t_i-t_{i-1})}}\mathbf{1}_{\{t_i\geq t_{i-1}\}} &= \frac{n!}{T_{n+1}^n e^{-\lambda T_{n+1}}}\prod_{i=1}^{n+1}e^{-\lambda(t_i-t_{i-1})}\mathbf{1}_{\{t_i\geq t_{i-1}\}} \\ &= \frac{n!}{T_{n+1}^n}\prod_{i=1}^{n+1}\mathbf{1}_{\{t_i\geq t_{i-1}\}} \end{aligned}$$

where $t_0=0$ and $t_{n+1}=T_{n+1}$. This is precisely the density of Q so we are done.

3. Consider a discrete time Markov chain X_n on a countable state space S. Define the hitting time $\tau_y=\inf\{n\geq 1: X_n=y\}$ and then define recursively $\tau_y^{k+1}=\tau_y^k+\tau_y\circ\theta_{\tau_y^k}$, where θ is the shift operator and $\tau_y^0=0$. Note that τ_y^k is the kth time the process visits y. Define also the occupation times

$$\kappa_y=\sup\{k:\tau_y^k<\infty\}=\sum_{n=1}^{\infty}\mathbf{1}\{X_n=y\}$$

for $y\in S$ and the following hitting probabilities

$$r_{xy} = P^x(\tau_y < \infty) = P^x(\kappa_y > 0).$$

a) In the discrete case, a random variable $T : \Omega \to \mathbb{N} \cup \{\infty\}$ is a stopping time if $\{\omega : T(\omega) = n\} \in \mathcal{F}_n$, where $\mathcal{F}_n$ is the filtration on the space. Argue that τ_y^k is a stopping time for each k.

b) Formulate and prove the strong Markov property in the discrete time and discrete space setup.

c) Show that

$$P^x(\kappa_y \geq k) = P^x(\tau_y^k < \infty) = r_{xy} r_{yy}^{k-1}.$$

d) Conclude that if the process starts from $x \in S$, the number of visits to x is either almost surely infinite, or almost surely finite. In the first case, the state x is called recurrent and in the second it is called transient.

Hints: a),b)$\Rightarrow$c)$\Rightarrow$d) and induction in c).

Solution: a) We shall do this by induction. The case $k = 0$ clearly holds. We note that we can write the event $\{\tau_y^{k+1} = t\}$ as

$$\begin{aligned}
\{\tau_y^{k+1} = t\} &= \bigcup_{s=0}^{t} \{\tau_y^k = s\} \cap \{\tau_y \circ \theta_{\tau_y^k} = t - s\} \\
&= \bigcup_{s=0}^{t} \{\tau_y^k = s\} \cap \{\tau_y \circ \theta_s = t - s\} \\
&= \bigcup_{s=0}^{t} \{\tau_y^k = s\} \cap \{\inf\{n \geq 1 : X_{s+n} = y\} = t - s\}.
\end{aligned}$$

The set $\{\inf\{n \geq 1 : X_{s+n} = y\} = t - s\}$ is completely determined by the chain up to time t, i.e. for a given ω, to know if $\inf\{n \geq 1 : X_{s+n}(\omega) = y\} = t - s$, we need to know only $(X_1(\omega), X_2(\omega), ..., X_t(\omega))$. Thus for $\mathcal{F}_j = \sigma(X_i, i \leq j)$, $\{\inf\{n \geq 1 : X_{s+n} = y\} = t - s\} \in \mathcal{F}_t$. On the other hand, by the induction assumption, $\{\tau_y^k = s\} \in \mathcal{F}_s$. Thus

$$\{\tau_y^k = s\} \cap \{\inf\{n \geq 1 : X_{s+n} = y\} = t - s\} \in \mathcal{F}_t$$

and $\{\tau_y^{k+1} = t\} \in \mathcal{F}_t$ so τ_y^k is a stopping time for each k.

b) One could formulate the strong Markov property just as in the continuum case. For the problem c), the following equivalent formulation is more useful. For a discrete time chain X and a discrete stopping time τ,

$$P(\theta_\tau X \in A | \mathcal{F}_\tau) = P^{X_\tau}(A)$$

almost surely on the set $\{\tau < \infty\}$ for any cylinder set A.

The proof of this goes as the continuous time one in the case where the stopping time takes only countably many values. For the proof of this, see the lectures.

c) Using the strong Markov property,

$$
\begin{aligned}
P^x(\tau_y^{k+1} < \infty) &= P^x(\tau_y^k < \infty, \tau_y \circ \theta_{\tau_y^k} < \infty) \\
&= P^x(\tau_y^k < \infty) P^y(\tau_y < \infty) \\
&= r_{yy} P^x(\tau_y^k < \infty).
\end{aligned}
$$

The result follows by induction.

d) So we see that if the process starts from x, the probability that it returns to x more than k times is given by

$$P^x(\kappa_x \geq k) = r_{xx}^k$$

If $r_{xx} = 1$, the process then almost surely returns to x infinitely many times. If $r_{xx} < 1$, it almost surely visits x only finitely many times.

4. a) Let X be a discrete time Markov process on some space S and let X have an invariant measure ν. Show that for any measurable $B \subset S$, $P^\nu(X_n \in B \text{ for infinitely many } n) \geq \nu(B)$.

b) Now let the space S be countable so that X is a discrete time Markov chain on S. Show that if $\nu(x) > 0$ for some state $x \in S$, then x is recurrent.

Hints: In b), use d) of the previous problem and a) from this one.

Solution: a) This is just the reverse Fatou lemma and the definition of an invariant measure:

$$
\begin{aligned}
P^\nu(X_n \in B \text{ for infinitely many } n) &= E^\nu\left(\limsup_{n\to\infty} \mathbf{1}_{\{X_n \in B\}}\right) \\
&\geq \limsup_{n\to\infty} P^\nu(X_n \in B) \\
&= \limsup_{n\to\infty} \nu(B) \\
&= \nu(B).
\end{aligned}
$$

b) By a) and using the definition of P^ν (i.e. $P^\nu(A) = \int P^x(A)\nu(dx)$)

$$
\begin{aligned}
&0 < \nu(x) \\
&\leq P^\nu(X_n = x \text{ for infinitely many } n) \\
&= \int P^y(X_n = x \text{ for infinitely many } n)\nu(dy).
\end{aligned}
$$

By problem 3 c), $P^y(X_n = x \text{ for infinitely many } n)$ can be non-zero only if $r_{xx} = 1$, so we see that $\nu(x) > 0$ implies that $r_{xx} = 1$ and x is recurrent.

5. let N_t be a Poisson process with rate λ and let $\mathcal{F}_t = \sigma(N_s, s \leq t)$ be the smallest σ-algebra so that N_s is $\mathcal{F}_t$ measurable when $s \leq t$. Show that

a)

$$E(N_t - \lambda t | \mathcal{F}_s) = N_s - \lambda s$$

b)

$$E((N_t - \lambda t)^2 - \lambda t | \mathcal{F}_s) = (N_s - \lambda s)^2 - \lambda s$$

Solution: a) We write

$$\begin{aligned} E(N_t - \lambda t | \mathcal{F}_s) &= E((N_t - N_s) - \lambda(t-s) | \mathcal{F}_s) + E(N_s - \lambda s | \mathcal{F}_s) \\ &= E(N_t - N_s - \lambda(t-s)) + N_s - \lambda s. \end{aligned}$$

Here we used the fact that $N_t - N_s$ is independent of $\mathcal{F}_s$ (i.e. that the Poisson process has independent increments) and that N_s is $\mathcal{F}_s$ measurable.

We now note that $N_t - N_s = N(s,t]$ is also poisson distributed and its mean is $\lambda(t-s)$ so we have the desired result.

b) Here we write

$$(N_t - \lambda t)^2 = (N_t - N_s - \lambda(t-s))^2 + 2(N_t - \lambda t)(N_s - \lambda s) - (N_s - \lambda s)^2.$$

Similar reasoning as in a) and using the result of a) yield

$$\begin{aligned} E((N_t - \lambda t)^2 - \lambda t | \mathcal{F}_s) &= E((N_t - N_s - \lambda(t-s))^2 - \lambda(t-s)) + 2(N_s - \lambda s)E(N_t - \lambda t | \mathcal{F}_s) \\ &\quad - (N_s - \lambda s)^2 - \lambda s \\ &= 0 + 2(N_s - \lambda s)^2 - (N_s - \lambda s)^2 - \lambda s \\ &= (N_s - \lambda s)^2 - \lambda s. \end{aligned}$$