STOCHASTIC PARTICLE SYSTEMS - EXERCISE 2, SOLUTIONS

1. a) Let $(\xi_n)_n$ be independent $\mathbb{R}_+$-valued random variables. Show that $\sum_n \xi_n$ converges almost surely if and only if $\sum_n E(\min(\xi_n, 1))$ converges.

Hint: For one direction, you might want to consider $E(e^{-\sum_n \xi_n})$ and use the inequality $1 - x \leq e^{-x} \leq 1 - ax$ with $a = 1 - e^{-1}$ and $x \in [0, 1]$ in some way.

b) Let X be a continuous time Markov chain on a countable state space S and let the process have a bounded rate function c. Let T_n be the time of the nth jump. Show that $T_n \to \infty$ a.s.

Hint: Kolmogorov's 0-1 law (see for example Durrett's Probability: Theory and examples or Kallenberg's Foundations of moderns probability) and a).

Solution: a) Assume that $\sum_n E(\min(\xi_n, 1)) < \infty$. Since ξ_n are non-negative, we can use Fubini's theorem and

$$E\left(\sum_n \min(\xi_n, 1)\right) = \sum_n E(\min(\xi_n, 1)) < \infty.$$

Thus $\sum_n \min(\xi_n, 1) < \infty$ a.s. This implies that almost surely $\xi_n > 1$ only for finitely many n (otherwise infinitely many terms in the series would be one and it would diverge). This implies that almost surely $\sum_n \xi_n$ and $\sum_n \min(\xi_n, 1)$ differ only for finitely many terms in the series. This does not affect the convergence so we see that also $\sum_n \xi_n < \infty$ a.s.

For the other direction, we note that if $\sum_n \xi_n < \infty$ a.s., then $\sum_n \min(\xi_n, 1)$ a.s., so we can assume that $\xi_n \leq 1$ for all n. Also for $a = 1 - e^{-1}$ and $x \in [0, 1]$, $1 - x \leq e^{-x} \leq 1 - ax$. Thus using independence

$$\begin{aligned}
0 < E&\left(\exp\left(-\sum_n \xi_n\right)\right) \\
&= \prod_n E(e^{-\xi_n}) \\
&\leq \prod_n (1 - aE(\xi_n)) \\
&\leq \prod_n e^{-aE(\xi_n)} \\
&= \exp\left(-a\sum_n E(\xi_n)\right)
\end{aligned}$$

so also $\sum_n E(\xi_n) < \infty$, which implies that in the general case, $\sum_n E(\min(1, \xi_n)) < \infty$.$\blacksquare$

b) We shall need the Kolmogorov 0-1 law:

Theorem: (Kolmogorov's 0-1 law) Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be independent σ-algebras such that $\mathcal{F}_n \subset \mathcal{F}$ for all n. Let $\mathcal{G}_n = \sigma(\mathcal{F}_n, \mathcal{F}_{n+1}, \ldots)$ be the smallest σ-algebra containing the σ-algebras $\mathcal{F}_m$ for $m \geq n$. Then for any $G \in \mathcal{G} = \cap_{n=1}^{\infty} \mathcal{G}_n$, $P(G) \in \{0, 1\}$.

For a proof see for example Durrett's Probability: Theory and examples or Kallenberg's Foundations of modern probability.

Now if we consider the setting of a) and set $\mathcal{F}_n = \sigma(\xi_n)$, we see that the event $\{\sum_n \xi_n \text{ converges}\}$ is a tail event, i.e. it is in the σ-algebra $\mathcal{G}$. By independence and Kolmogorov's 0-1 law, we see that this event has either probability one or zero. Thus its complement, the event that the series diverges has either probability zero or one and the probability of these events is determined by the summability of $E(\min(\xi_n, 1))$.

Considering now the Markov chain, we note that T_n can be bounded from below by a sum of independent non-negative random variables: $T_n = \sum_{k=0}^{n-1} \frac{\tau_k}{c(Y_k)}$, where (Y_n) is the discrete time Markov chain used to build X and τ_n is exponential with mean one. So if c is bounded, say $c(x) \leq M$ for all x, then $T_n \geq M^{-1} \sum_{k=0}^{n-1} \tau_k$. Also $E(\min(\tau_n, 1)) = m > 0$, where m is independent of n so our reasoning implies that $\sum_{k=0}^{n-1} \tau_k \to \infty$ a.s. so $T_n \to \infty$ a.s.

Remark: One could of course use the law of large numbers in b).

2. Consider a continuous time Markov chain X (on some countable state space S) with transition probability $p_t(x,y) = P^x(X_t = y)$ and some bounded strictly positive rate function. Show that it has the Markov property, i.e. that for any $0 \leq t_0 < t_1 < ... < t_n$ and $x, x_0, ..., x_n \in S$,

$$P^x(X_{t_n} = x_n | X_{t_{n-1}} = x_{n-1}, ..., X_{t_0} = x_0) = p_{t_n - t_{n-1}}(x_{n-1}, x_n)$$

whenever the event we condition on has positive probability.

Hint: To keep things simpler (mainly notation), consider first the $n = 1$ case.

Solution: By the previous problem, $T_n \to \infty$ a.s. so we can write

$$\begin{aligned}
p_t(x,y) &= \sum_{n=0}^{\infty} P^x(X_t = y; T_n \leq t < T_{n+1}) \\
&= \sum_{n=0}^{\infty} P^x(Y_n = y; T_n \leq t < T_{n+1}) \\
&= \sum_{n=0}^{\infty} \sum_{y_1,...,y_{n-1} \in S} P^x(T_n \leq t < T_{n+1} | Y_n = y, Y_{n-1} = y_{n-1}, ..., Y_1 = y_1) \\
&\times P^x(Y_n = y, Y_{n-1} = y_{n-1}, ..., Y_1 = y_1).
\end{aligned}$$

If any of the events we condition on has zero probability, we interpret that term in the sum as zero. If we write $y_0 = x$ and $y_n = y$, we see that

$$P^x(T_n \leq t < T_{n+1} | Y_n = y, Y_{n-1} = y_{n-1}, ..., Y_1 = y_1) = P\left(\sum_{k=0}^{n-1} \tau_k c(y_k)^{-1} \leq t < \sum_{k=0}^{n} \tau_k c(y_k)^{-1}\right).$$

If we write $Q_{\{y_0,...,y_{n-1}\}}$ for the law of $\sum_{k=0}^{n-1} \tau_k c(y_k)^{-1}$, we see by Fubini's theorem that the fact that $\tau_n c(y)^{-1}$ is exponential implies that

$$p_t(x,y) = \sum_{n=0}^{\infty} \sum_{y_1,...,y_{n-1}\in S} \int_0^t e^{-c(y)(t-s)} dQ_{\{y_0,...,y_{n-1}\}}(s) P^x(Y_1 = y_1, ..., Y_n = y).$$

Let us now show that for $t_1 < t_2$ and $x_1, x_2 \in S$,

$$P^x(X_{t_1} = x_1, X_{t_2} = x_2) = P^x(X_{t_1} = x_1) P^{x_1}(X_{t_2 - t_1} = x_2).$$

We begin with the following expansion

$$\begin{aligned} &P^x(X_{t_1} = x_1, X_{t_2} = x_2) \\ &\quad = \sum_{0\le n_1 \le n_2} P^x(Y_{n_1} = x_1, Y_{n_2} = x_2, T_{n_1} \le t_1 < T_{n_1} + \tau_{n_1} c(x_1)^{-1}, T_{n_2} \le t_2 < T_{n_2} + \tau_{n_2} c(x_2)^{-1}), \end{aligned}$$

which is just using the fact that the intervals $[T_n, T_{n+1})$ partition $[0,\infty)$ and we impose $t_1 < t_2$ in the summation. Let us now condition/sum over all of the states Y_i for $i = 1, ..., n_2$, while imposing $Y_{n_1} = x_1$ and $Y_{n_2} = x_2$ with a Kroenecker δ inside the sum. We have

$$\begin{aligned} &P^x(X_{t_1} = x_1, X_{t_2} = x_2) \\ &\quad = \sum_{0\le n_1\le n_2} \sum_{y_1,...,y_{n_2}\in S} \delta_{y_{n_1},x_1} \delta_{y_{n_2},x_2} P^x(T_{n_j} \le t_j < T_{n_j} + \tau_{n_j} c(x_j)^{-1}, j = 1,2 | Y_k = y_k, k = 1, ..., n_2) \\ &\quad \times P^x(Y_k = y_k, k = 1, ..., n_2). \end{aligned}$$

Since $\tau_{n_1} c(x_1)^{-1}$ is exponential with parameter $c(x_1)$, this becomes

$$\begin{aligned} &\sum_{0\le n_1\le n_2} \sum_{y_1,...,y_{n_2}\in S} \delta_{y_{n_1},x_1} \delta_{y_{n_2},x_2} \int_{u_1\le t_1} \int_{t_1 < u_1+u_2 < t_2} c(x_1) e^{-c(x_1)u_2} \\ &\times P^x\left(u_1 + u_2 + \sum_{l=n_1+1}^{n_2-1} \tau_l c(y_l) \le t_2 < u_1 + u_2 + \sum_{l=n_1+1}^{n_2} \tau_l c(y_l)\right) du_2 dQ_{\{y_0,...,n_1-1\}}(u_1) \\ &\quad \times P^x(Y_k = y_k, k = 1, ..., n_2). \end{aligned}$$

Next we make the change of variables $s_1 = u_1$ and $s_2 = u_1 + u_2 - t_1$. Then we have

$$\begin{aligned} &\sum_{0\le n_1\le n_2} \sum_{y_1,...,y_{n_2}\in S} \delta_{y_{n_1},x_1} \delta_{y_{n_2},x_2} \int_{s_1=0}^{t_1} \int_{s_2=0}^{t_2-t_1} c(x_1) e^{-c(x_1)(s_2 - s_1 + t_1)} \\ &\times P^x\left(s_2 + \sum_{l=n_1+1}^{n_2-1} \tau_l c(y_l) \le t_2 - t_1 < s_2 + \sum_{l=n_1+1}^{n_2} \tau_l c(y_l)\right) ds_2 dQ_{\{y_0,...,n_1-1\}}(s_1) \\ &\quad \times P^x(Y_k = y_k, k = 1, ..., n_2). \end{aligned}$$

Now using the Markov property of the chain (Y_n), we see that this factors nicely:

$$\sum_{0\le n_1\le n_2}\sum_{y_1,\dots,y_{n_1}\in S}\delta_{y_{n_1},x_1}\int_{s_1=0}^{t_1}e^{-c(x_1)(t_1-s_1)}dQ_{\{y_0,\dots,n_1-1\}}(s_1)P^x(Y_i=y_i,i=1,\dots,n_1)$$

$$\times\sum_{y_{n_1+1},\dots,y_{n_2}\in S}\delta_{y_{n_2},x_2}\int_{s_2=0}^{t_2-t_1}P\left(s_2+\sum_{l=n_1+1}^{n_2-1}\tau_l c(y_l)^{-1}\le t_2-t_1<s_2+\sum_{l=n_1+1}^{n_2}\tau_l c(y_l)^{-1}\right)c(x_1)e^{-c(x_1)s_2}ds_2$$

$$\times P^{x_1}(Y_1=y_{n_1+1},\dots,Y_{n_2-n_1}=y_{n_2}).$$

We now set $m=n_2-n_1$ and split the sum $\sum_{n_1\le n_2}$ into $\sum_{n_1\ge 0}\sum_{m\ge 0}$. We note after renaming the summation variables, the second sum then becomes independent of n_1 and we can write it as

$$\sum_{m=0}^{\infty}\sum_{z_1,\dots,z_m\in S}\delta_{z_m,x_2}P^{x_1}\left(\sum_{l=0}^{m-1}\tau_l c(z_l)^{-1}\le t_2-t_1<\sum_{l=0}^{m}\tau_l c(z_l)^{-1}\right)P^{x_1}(Y_1=z_1,\dots,Y_m=z_m)$$
$$p_{t_2-t_1}(x_1,x_2),$$

where we wrote $z_0=x_1$. The n_1 sum we recognize as $p_{t_1}(x,x_1)$. So we conclude that

$$P^x(X(t_1)=x_1,X(t_2)=x_2)=p_{t_1}(x,x_1)p_{t_2-t_1}(x_1,x_2).$$

Looking at our proof up to now, we notice that the same reasoning works for proving

$$P^x(X(t_1)=x_1,X(t_2)=x_2,\dots,X(t_n)=x_n)=p_{t_1}(x,x_1)p_{t_2-t_1}(x_1,x_2)\cdot\ldots\cdot p_{t_n-t_{n-1}}(x_{n-1},x_n).$$

The only difference in practice would be more indeces n_i, but again we would see the same factoring and we would be able to use the Markov property of the discrete chain etc. So in fact we have proved the Markov property for X.

3. Consider a metric space (Y,d). Let $q:Y\times Y\to\mathbb{R}_+$, $q(x,y)=\min(d(x,y),1)$ (this is also a metric on Y) and let

$$\Lambda'=\{\lambda\in C(\mathbb{R}_+,\mathbb{R}_+):\lambda\text{ is an increasing bijection}\}.$$

Then define

$$\gamma(\lambda)=\sup_{s>t\ge 0}\left|\log\frac{\lambda(s)-\lambda(t)}{t-s}\right|$$

and $\Lambda=\{\lambda\in\Lambda':\gamma(\lambda)<\infty\}$.

For $x,y\in D_Y[0,\infty)$ (the space of functions from $[0,\infty)$ into Y which are right continuous and have left limits) and $\lambda\in\Lambda$ we define

$$\rho_s(x,y,\lambda)=\sup_{t\ge 0}q(x(\min(t,s)),y(\min(\lambda(t),s)))$$

and

$$\rho(x,y) = \inf_{\lambda\in\Lambda}\left\{\max\left(\gamma(\lambda), \int_0^\infty e^{-s}\rho_s(x,y,\lambda)ds\right)\right\}.$$

a) Show that ρ is a metric on $D_Y[0,\infty)$ (sometimes called the Skorohod metric and its topology is called the Skorohod topology on $D_Y[0,\infty)$).

b) Let (x_n) be a sequence in $D_Y[0,\infty)$ and $x \in D_Y[0,\infty)$. Show that $\rho(x_n, x) \to 0$ if and only if there is a sequence (λ_n) in Λ so that $\gamma(\lambda_n) \to 0$ and $\rho_s(x_n, x, \lambda_n) \to 0$ at all points s where x is continuous.

Hint: In b) you'll probably need to use the result that each $x \in D_Y[0,\infty)$ has only countably many points of discontinuity.

Solution: a) We begin by noting that

$$\begin{aligned}\rho_s(x,y,\lambda) &= \sup_{t\geq 0} q(x(\min(t,s)), y(\min(\lambda(t),s)))\\ &= \sup_{t\geq 0} q(x(\min(\lambda^{-1}(t),s)), y(\min(t,s)))\\ &= \rho_s(y,x,\lambda^{-1}).\end{aligned}$$

Moreover, we note that $\gamma(\lambda) = \gamma(\lambda^{-1})$. These remarks imply that $\rho(x,y) = \rho(y,x)$.

Let us now assume that $\rho(x,y) = 0$. To show that $x = y$, the right continuity of the functions implies that it is enough to check that they agree on points where they are continuous. To do this, we note that $\rho(x,y) = 0$ implies that there is some sequence $\lambda_n \in \Lambda$ so that $\gamma(\lambda_n) \to 0$ and

$$\int_0^\infty e^{-s}\rho_s(x,y,\lambda_n)ds \to 0.$$

Thus for any $\epsilon > 0$ and $s_0 > 0$

$$m\left(\{s \in [0,s_0] : \rho_s(x,y,\lambda_n) \geq \epsilon\}\right) \to 0,$$

where m is the Lebesgue measure on $\mathbb{R}$. Since $\gamma(\lambda_n) \to 0$, we also have that for any $\epsilon > 0$, there exists a N_ϵ so that for any $s > t \geq 0$ and $n \geq N_\epsilon$,

$$\left|\log\frac{\lambda_n(s)-\lambda_n(t)}{s-t}\right| < \epsilon$$

so some manipulation shows that for $n \geq N_\epsilon$ and $s \in (0, s_0]$,

$$s_0(e^{-\epsilon}-1) < \lambda_n(s) - s < (e^\epsilon - 1)s_0,$$

which implies that for any $T > 0$,

$$\lim_{n\to\infty}\sup_{0\leq t\leq T}|\lambda_n(t) - t| = 0.$$

Let us now consider a point t so that x and y are continous at t and $x(t) \neq y(t)$. Let $\epsilon > 0$ be so that $d(x(t), y(t)) > \epsilon$. By continuity, there is a neighbourhood of t so that $d(x(s), y(s)) > \epsilon$ in this neighborhood. By the uniform convergence of λ_n, this implies that there is an interval so that for large enough n, $\rho_s(x,y,\lambda_n) \geq \epsilon$. Since open intervals have positive measure, this contradicts our previous result. So x and y must agree on points where they are continuous and since we are considering right continuous functions, $x = y$.

We still need to show the triangular inequality. For this, we note that for $x, y, z \in D_Y[0,\infty)$ and $\lambda_1, \lambda_2 \in \Lambda$,

$$\begin{aligned}\sup_{s\geq 0} q(x(\min(t,s)), z(\min(\lambda_2(\lambda_1(t)),s))) &\leq \sup_{s\geq 0} q(x(\min(t,s)), y(\min(\lambda_1(t),s)))\\ &\quad + \sup_{s\geq 0} q(y(\min(\lambda_1(t),s)), z(\min(\lambda_2(\lambda_1(t)),s)))\\ &= \sup_{s\geq 0} q(x(\min(t,s)), y(\min(\lambda_1(t),s)))\\ &\quad + \sup_{s\geq 0} q(y(\min(t,s)), z(\min(\lambda_2(t),s))).\end{aligned}$$

Thus $\rho_s(x,z,\lambda_2\circ\lambda_1) \leq \rho_s(x,y,\lambda_1) + \rho_s(y,z,\lambda_2)$. We also note that

$$\begin{aligned}\sup_{s>t\geq 0}\left|\log\frac{\lambda_2(\lambda_1(s)) - \lambda_2(\lambda_1(t))}{s-t}\right| &= \sup_{s>t\geq 0}\left|\log\frac{\lambda_2(\lambda_1(s)) - \lambda_2(\lambda_1(t))}{\lambda_1(s)-\lambda_1(t)} + \log\frac{\lambda_1(s)-\lambda_1(t)}{s-t}\right|\\ &\leq \gamma(\lambda_2) + \gamma(\lambda_1).\end{aligned}$$

So $\lambda_2\circ\lambda_1 \in \Lambda$ and we conclude that $d(x,z) \leq d(x,y) + d(y,z)$.

b) Since $\rho_s(x,y,\lambda) \leq 1$, by dominated convergence and using the fact that functions in $D_Y[0,\infty)$ have only countably many points of discontinuity we see that $\rho_s(x_n,x,\lambda_n) \to 0$ on the continuity points of x implies that $\int_0^\infty e^{-s}\rho_s(x_n,x,\lambda_n)ds \to 0$ so we see that if in addition $\gamma(\lambda_n) \to 0$, then $\rho(x,y) \to 0$.

For the other direction, let $\rho(x_n,x) \to 0$ and let t_0 be a point of continuity for x. As in a), $\rho(x_n,x) \to 0$ implies that there is a sequence λ_n with $\gamma(\lambda_n) \to 0$ so that for any fixed $s_0 > 0$ and $\epsilon > 0$,

$$m\left(\{s \in [0,s_0] : \rho_s(x_n,x,\lambda_n) \geq \epsilon\}\right) \to 0.$$

In particular, we can pick points $t_n > t_0$ so that

$$\rho_{t_n}(x_n,x,\lambda_n) = \sup_{t\geq 0} q(x_n(\min(t,t_n)), x(\min(\lambda_n(t),t_n))) \to 0.$$

Using the triangular inequality of q, we have

$$\begin{aligned}\sup_{t\geq 0} q(x_n(\min(t,t_0)), x(\min(\lambda_n(t),t_0))) &\leq \sup_{t\geq 0} q(x_n(\min(t,t_0)), x(\min(\lambda_n(\min(t,t_0)),t_n)))\\ &\quad + \sup_{t\geq 0} q(x(\min(\lambda_n(\min(t,t_0)),t_n)), x(\min(\lambda_n(t),t_0))).\end{aligned}$$

We note that since $t_0 < t_n$ for all n, we can write the first term as

$$\sup_{t\in[0,t_0]} q(x_n(\min(t,t_n)), x(\min(\lambda_n(t),t_n)) \to 0$$

where the convergence follows from the definition of the points t_n. For the second term, consider the quantity we are maximizing for $t \geq t_0$ and $t < t_0$ separately. For $t \geq t_0$ we have

$$q(x(\min(\lambda_n(\min(t, t_0)), t_n)), x(\min(\lambda_n(t), t_0))) = q(x(\min(\lambda_n(t_0), t_n)), x(\min(\lambda_n(t), t_0))).$$

We recall that $\lambda_n(t) \to t$ uniformly on compact sets so $\min(\lambda_n(t_0), t_n) \to t_0$ since $t_n > t_0$ for all n. Also $\min(\lambda_n(t), t_0) \to t_0$ uniformly in $t \geq t_0$ so continuity of x at t_0 that for $t \geq t_0$ we have uniform convergence to 0.

For $t < t_0$,

$$q(x(\min(\lambda_n(\min(t, t_0)), t_n)), x(\min(\lambda_n(t), t_0))) = q(x(\min(\lambda_n(t), t_n)), x(\min(\lambda_n(t), t_0)))$$

and one can again use the uniform convergence of λ_n and continuity of x at t_0 to deduce that we have uniform convergence of this to zero.

Thus we have found a sequence λ_n so that for each point of continuity t_0, $\rho_{t_0}(x, x_n, \lambda_n) \to 0$.

Remark: In a) we did not explicitly use the fact that a right continuous function with left limits has only countably many points of discontinuity. So the hint in the initial version of the problem was misleading.

4. Let X be a continuous time Markov chain on a countable space S. We can interpret X as a mapping $X : \Omega \to D_S[0, \infty)$, where Ω is the product of the sample space of a discrete time Markov chain and the sample space of a sequence of i.i.d. random times with exponential distribution and mean one. Let Σ be the product σ-algebra on Ω (the product of the discrete time Markov chain σ-algebra and the σ-algebra of the i.i.d. exponential times) and $\mathcal{F}$ be the σ-algebra generated by the cylinder sets of $D_S[0, \infty)$. Show that $X : (\Omega, \Sigma) \to (D_S[0, \infty), \mathcal{F})$ is measurable.

Solution: We begin by noting that a mapping between measurable spaces $f : (X, \mathcal{F}) \to (Y, \mathcal{G})$ is measurable if $f^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{C} \subset \mathcal{G}$ where $\mathcal{C}$ generates the σ-algebra $\mathcal{G}$, i.e. $\sigma(\mathcal{C}) = \mathcal{G}$. Let $\mathcal{T} \subset 2^Y$ be the class of sets $A \subset Y$ so that $f^{-1}(A) \in \mathcal{F}$. Then using elementary properties of the inverse image of a function, one can check that $\mathcal{T}$ is a σ-algebra. By assumption, $\mathcal{C} \subset \mathcal{T}$. Thus, $\mathcal{G} = \sigma(\mathcal{C}) \subset \sigma(\mathcal{T}) = \mathcal{T}$ so $f^{-1}(A) \in \mathcal{F}$ for all $A \in \mathcal{G}$ as well, i.e. f is measurable.

Since we are dealing with the cylinder σ-algebra, which in the case of a countable state space is generated by sets of the form $\{\xi \in D_S[0, \infty) : \xi(t) = y\}$, i.e. in our formulation above, $\mathcal{G}$ is the cylinder σ-algebra and

$$\mathcal{C} = \{\{\xi \in D_S[0, \infty) : \xi_t = y\} : t \in [0, \infty) \text{ and } y \in S\ \}.$$

By our initial remark, to show that X is measurable, we need to show that for each $t \geq 0$ and $y \in S$,

$$\{\omega \in \Omega : X_t(\omega) = y\} \in \Sigma.$$

By the definition of the process X, we can write

$$\{\omega \in \Omega : X_t(\omega) = y\} = \bigcup_{n=0}^{\infty} \{\omega \,| T_n(\omega) \leq t < T_{n+1}(\omega), Y_n(\omega) = y\} \,.$$

The sets $\{\omega | T_n(\omega) \leq t < T_{n+1}(\omega), Y_n(\omega) = y\}$ are certainly in Σ so their countable union is as well and we see that X is measurable.

5. Let S be a countable abelian group (stick to $S = \mathbb{Z}^d$ if you want to) and $p_t(x, y)$ be a transition probability which is symmetric and translation invariant: $p_t(x, y) = p_t(y, x) = p_t(0, y - x)$. Let X and Y be independent identically distributed continuous time Markov cahins on S with transition probabilities p_t. Show that $Z = X - Y$ has the same distribution of $(X_{2t})_{t \geq 0}$, i.e. the process run at double speed.

Hint: It is enough to check that the finite dimensional distributions agree, i.e. you'll probably want to show that

$$P^{x,y}(Z_{t_1} = z_1, ..., Z_{t_n} = z_n)$$

factors nicely like a Markov chain should, only depends on $x - y$ and agrees with the corresponding quantity for $(X_{2t})_t$. To check that it really is enough that finite dimensional distributions agree, you'll need a monotone class argument (or sometimes called Dynkin's π-λ theorem) or just look it up in any good book on probability.

Solution: Let $0 \leq t_1 < t_2 < ... < t_n$ and $x, y, z_1, ..., z_n \in S$. Then by independence the independence of X and Y and using the fact that they are both

$$\begin{aligned}
&P^{x,y}(X_{t_1} - Y_{t_1} = z_1, ..., X_{t_n} - Y_{t_n} = z_n) \\
&\quad = \sum_{\alpha_1,...,\alpha_n \in S} P^{x,y}(X_{t_1} = \alpha_1, Y_{t_1} = z_1 - \alpha_1, ..., X_{t_n} = \alpha_n, Y_{t_n} = z_n - \alpha_n) \\
&\quad = \sum_{\alpha_1,...,\alpha_n \in S} P^{x}(X_{t_1} = \alpha_1, ..., X_{t_n} = \alpha_n) P^{y}(Y_{t_1} = z_1 - \alpha_1, ..., Y_{t_n} = z_n - \alpha_n) \\
&\quad = \sum_{\alpha_1,...,\alpha_n \in S} p_{t_1}(x, \alpha_1) \cdot ... p_{t_n - t_{n-1}}(\alpha_{n-1}, \alpha_n) p_{t_1}(y, z_1 - \alpha_1) \cdot ... p_{t_n - t_{n-1}}(z_{n-1} - \alpha_{n-1}, z_n - \alpha_n) \\
&\quad = \sum_{\alpha_1 \in S} p_{t_1}(x, \alpha_1) p_{t_1}(y, z_1 - \alpha_1) \cdot ... \cdot \sum_{\alpha_n \in S} p_{t_n - t_{n-1}}(\alpha_{n-1}, \alpha_n) p_{t_n - t_{n-1}}(z_{n-1} - \alpha_{n-1}, z_n - \alpha_n).
\end{aligned}$$

We now note by the symmetry and translational invariance of p that

$$\begin{aligned}
&\sum_{\alpha_n \in S} p_{t_n - t_{n-1}}(\alpha_{n-1}, \alpha_n) p_{t_n - t_{n-1}}(z_{n-1} - \alpha_{n-1}, z_n - \alpha_n) \\
&\quad = \sum_{\alpha_n \in S} p_{t_n - t_{n-1}}(\alpha_{n-1}, \alpha_n) p_{t_n - t_{n-1}}(\alpha_n, z_n - z_{n-1} + \alpha_{n-1}) \\
&\quad = p_{2(t_n - t_{n-1})}(\alpha_{n-1}, z_n - z_{n-1} + \alpha_{n-1}) \\
&\quad = p_{2(t_n - t_{n-1})}(z_{n-1}, z_n).
\end{aligned}$$

A similar calculation for the α_1 sum shows that

$$P^{x,y}(X_{t_1}-Y_{t_1}=z_1,...,X_{t_n}-Y_{t_n}=z_n)=p_{2t_1}(x-y,z_1)p_{2(t_2-t_1)}(z_1,z_2)\cdot ...\cdot p_{2(t_n-t_{n-1})}(z_{n-1},z_n).$$

So (assuming the event we condition on has positive probability)

$$P^{x,y}(X_{t_n}-Y_{t_n}=z_n|X_{t_1}-Y_{t_1}=z_1,...,X_{t_{n-1}}-Y_{t_{n-1}}=z_{n-1})=p_{2(t_n-t_{n-1})}(z_{n-1},z_n).$$

As we saw earlier, the finite dimensional distributions of $X-Y$ depend only on the difference of the points X_0, Y_0 and not their actual values. We conclude that the process X_{2t} agrees with the process $X_t - Y_t$ at least on the level on finite dimensional distributions. To see that they in fact agree in distribution, we shall need the monotone class theorem.

Theorem. (*Definitions*) Let Ω be a set. A collection $\mathcal{C}\subset 2^\Omega$ is called a π-system if $A, B\in\mathcal{C}$ implies that $A\cap B\in\mathcal{C}$. A collection $\mathcal{C}\subset 2^\Omega$ is called a λ-system if $\Omega\in\mathcal{D}$, $A, B\in\mathcal{D}$ and $A\subset B$ implies that $B\setminus A\in\mathcal{D}$ and $A_1, A_2, ...\in\mathcal{D}$ so that $A_1\subset A_2\subset ...$ implies that $\cup_{n=1}^\infty A_n\in\mathcal{D}$.

(*Statement*) If $\mathcal{C}$ is a π-system, $\mathcal{D}$ is a λ-system and $\mathcal{C}\subset\mathcal{D}$, then $\sigma(\mathcal{C})\subset\mathcal{D}$.

For a proof, see any good book on probability (e.g. Kallenberg).

Using this, we can show that the finite dimensional distributions are enough.

Proposition. Let X, Y be processes on an index set T (in our case $T=[0,\infty)$) with paths in $U\subset S^T$ (so here S is the space where the processes takes values so their paths are elements of S^T). Then the distributions of X and Y agree if the distributions of $(X_{t_1}, ..., X_{t_n})$ and $(Y_{t_1}, ..., Y_{t_n})$ agree for all n and $t_i\in T$.

Proof: Let $\mathcal{S}$ be the σ-algebra of S and $\mathcal{S}^T$ be the cylinder σ-algebra on S^T. Denote by $\mathcal{D}$ be the collection of sets $A\in\mathcal{S}^T$ so that $P(X\in A)=P(Y\in A)$. Also let $\mathcal{C}$ be the collection of sets

$$A=\{f\in S^T : (f_{t_1}, ..., f_{t_n})\in B\}$$

where n runs over $\mathbb{N}$, t_i over T and B over $\mathcal{S}^n$ (the n-fold product σ-algebra of $\mathcal{S}$). Now $\mathcal{C}$ is a π-system and some basic measure theory implies that $\mathcal{D}$ is a λ-system. By definition, $\sigma(\mathcal{C})=\mathcal{S}^T$ so the monotone class theorem implies that $\mathcal{S}^T\subset\mathcal{D}$. Thus we conclude that $P(X\in A)=P(Y\in A)$ for all sets A in the cylinder σ-algebra so the distributions of X and Y agree.

We conclude that $(X_t-Y_t)_t$ and $(X_{2t})_t$ have the same distributions.

Remarks: The original formulation of the problem was incorrect. In general, if a discrete time transition probability is symmetric, a continuous time transition probability need not be symmetric. In fact, as argued in Seppäläinen's notes

$$\left.\frac{d}{dt}\right|_{t=0} p_t(x,y)=c(x)p(x,y),$$

where c is the rate function. So if p_t is symmetric, then a necessary condition would be that $c(x)p(x,y)=c(y)p(y,x)$. In fact, using semi-group theory one can show that in our case p_t is symmetric and translation invariant if and only if $q(x,y)=c(x)p(x,y)$ is as well (here p is the transition probability for the discrete time chain).