STOCHASTIC PARTICLE SYSTEMS: EXERCISE 9

Recall the Dirichlet norm considered in the lectures:

$$D(g) = \sum_{u,v:|u-v|=1} \int (g(\eta^{u,v}) - g(\eta))^2 \nu(d\eta),$$

where ν is a Bernoulli measure. Recall also the 'Dirichlet norm' we defined by restricting to subsets. Here we defined for any cube $B \subset \Lambda_N$ the marginals of f on B and B^C:

$$f_B(\eta) = \int f(\eta, \eta^C)\nu(d\eta^C),$$

where $\eta \in \{0,1\}^B$ and $\eta^C \in \{0,1\}^{B^C}$ (so we average out the degrees of freedom related to B^C). We then define

$$D_B(f_B) = \sum_{u,v\in B:|u-v|=1} \int (f_B(\eta^{u,v}) - f_B(\eta))^2 \nu(d\eta)$$

and a similar definition for $D_{B^C}(f_{B^C})$.

1. Show that the Dirichelt norm is subadditive: $D(\sqrt{f}) \geq D_B(\sqrt{f_B}) + D_{B^C}(\sqrt{f_{B^C}})$.

2. Show that if $D_B(\sqrt{f}) = 0$, then $\sqrt{f}(\eta) = g\left(\sum_{x\in B} \eta(x)\right)$, i.e. that if the Dirichlet norm of a function vanishes, the function is actually only a function of the number of particles and not of the actual configuration.

Recall that we denoted by μ_n the Bernoulli measure conditioned to have n particles present in $\overline{B}_1$ recall that B_1 was a cube of side length M and $\overline{B}_1$ was a cube of sidelentgh $M + 2R$. We also denoted by $\langle\cdot\rangle_B$ a spatial average over the set B.

3. Show that there is some positive constant C so that

$$\mathbb{E}((\langle h\rangle_{B_1} - \mathbb{E}_{\mu_n}(h))^2) \leq \frac{C}{M^d}.$$

4. Show that

$$\max_n \int |\mathbb{E}_{\mu_n}(h) - \mathbb{E}_{\nu_{\frac{n}{p}}}(h)| d\mu_n \to 0$$

as $M \to \infty$. Here $p = (M + 2R)^d$, i.e. the volume of $\overline{B}_1$.