STOCHASTIC PARTICLE SYSTEMS - EXERCISE 3

1. Let $(\mathcal{T}_i)_{i=1}^{\infty}$ be independent Poisson processes with rates $(r_i)_{i=1}^{\infty}$ satisfying $\sum_i r_i < \infty$. Let $\mathcal{T} = \cup_i \mathcal{T}_i$. Show that $\mathcal{T}$ is a Poisson process with rate r. Moreover, for any $s \in (0, \infty)$, show that the first point in $\mathcal{T}$ that is larger than s comes from the set $\mathcal{T}_i$ with probability $\frac{r_i}{r}$.

2. Let $\mathcal{T} = \{T_1, T_2, ...\}$ be a Poisson process. Show that given T_{n+1}, the distribution of $(T_1, ..., T_n)$ is the same as the distribution of $(S_1, ..., S_n)$, where $(S_1, ..., S_n)$ is gotten from i.i.d uniform random variables on $[0, T_{n+1}]$ by putting the random variables into increasing order.

3. Consider a discrete time Markov chain X_n on a countable state space S. Define the hitting time $\tau_y = \inf\{n \in \mathbb{N} : X_n = y\}$ and then define recursively $\tau_y^{k+1} = \tau_y^k + \tau_y \circ \theta_{\tau_y^k}$, where θ is the shift operator and $\tau_y^0 = 0$. Note that τ_y^k is the kth time the process visits y. Define also the occupation times

$$\kappa_y = \sup\{k : \tau_y^k < \infty\} = \sum_{n=0}^{\infty} \mathbf{1}\{X_n = y\}$$

for $y \in S$ and the following hitting probabilities

$$r_{xy} = P^x(\tau_y < \infty) = P^x(\kappa_y > 0).$$

a) In the discrete case, a random variable $T : \Omega \to \mathbb{N}$ is a stopping time if $\{\omega : T(\omega) = n\} \in \mathcal{F}_n$, where $\mathcal{F}_n$ is the filtration on the space. Argue that τ_y^k is a stopping time for each k.

b) Formulate and prove the strong Markov property in the discrete time and discrete space setup.

c) Show that

$$P^x(\kappa_y \geq k) = P^x(\tau_y^k < \infty) = r_{xy} r_{yy}^{k-1}.$$

d) Conclude that if the process starts from $x \in S$, the number of visits to x is either almost surely infinite, or almost surely finite. In the first case, the state x is called recurrent and in the second it is called transient.

Hints: a)⇒b)⇒c)⇒d) and induction in c).

4. a) Let X be a discrete time Markov process on some space S and let X have an invariant measure ν. Show that for any measurable $B \subset S$, $P^\nu(X_n \in B$ for infinitely many $n) \geq \nu(B)$.

b) Now let the space S be countable so that X is a discrete time Markov chain on S. Show that if $\nu(x) > 0$ for some state $x \in S$, then x is recurrent.

Hints: In b), use d) of the previous problem and a) from this one.

5. Let N_t be a Poisson process of rate λ and let $\mathcal{F}_t = \sigma(N_s, s \leq t)$ be the smallest σ-algebra so that N_s is $\mathcal{F}_t$ measurable when $s \leq t$. Show that

a)

$$E(N_t - \lambda t | \mathcal{F}_s) = N_s - \lambda s.$$

b)

$$E((N_t - \lambda t)^2 - \lambda t | \mathcal{F}_s) = (N_s - \lambda s)^2 - \lambda s.$$