STOCHASTIC PARTICLE SYSTEMS - EXERCISE 2

1. a) Let $(\xi_n)_n$ be independent $\mathbb{R}_+$-valued random variables. Show that $\sum_n \xi_n$ converges almost surely if and only if $\sum_n E(\min(\xi_n, 1))$ converges.

Hint: For one direction, you might want to consider $E(e^{-\sum_n \xi_n})$ and use the inequality $1 - x \leq e^{-x} \leq 1 - ax$ with $a = 1 - e^{-1}$ and $x \in [0,1]$ in some way.

b) Let X be a continuous time Markov chain on a countable state space S and let the process have a bounded rate function c. Let T_n be the time of the nth jump. Show that $T_n \to \infty$ a.s.

Hint: Kolmogorov's 0-1 law (see for example Durrett's Probability: Theory and examples or Kallenberg's Foundations of modern probability) and a).

2. Consider a continuous time Markov chain X (on some countable state space S) with transition probability $p_t(x, y) = P^x(X_t = y)$ and some bounded rate function c. Show that it has the Markov property, i.e. that for any $0 \leq t_0 < t_1 < ... < t_n$ and $x, x_0, ..., x_n \in S$,

$$P^x(X_{t_n} = x_n | X_{t_{n-1}} = x_{n-1}, ..., x_{t_0} = x_0) = p_{t_n - t_{n-1}}(x_{n-1}, x_n)$$

whenever the event we condition on has positive probability.

Hint: To keep things simpler (mainly notation), consider first the $n = 1$ case. Also you might want to use $P(A) = \sum_n P(A; T_n \leq t < T_{n+1})$.

3. Consider a metric space (Y, d). Let $q : Y \times Y \to \mathbb{R}_+$, $q(x, y) = \min(d(x, y), 1)$ (this is also a metric on Y) and let

$$\Lambda' = \{\lambda \in C(\mathbb{R}_+, \mathbb{R}_+) : \lambda \text{ is an increasing bijection}\}.$$

Then define

$$\gamma(\lambda) = \sup_{s > t \geq 0} \left| \log \frac{\lambda(s) - \lambda(t)}{s - t} \right|$$

and $\Lambda = \{\lambda \in \Lambda' : \gamma(\lambda) < \infty\}$.

For $x, y \in D_Y[0, \infty)$ (the space of functions from $[0, \infty)$ into Y which are right continuous and have left limits) and $\lambda \in \Lambda$ we define

$$\rho_s(x, y, \lambda) = \sup_{t \geq 0} q(x(\min(t, s)), y(\min(\lambda(t), s)))$$

and

$$\rho(x, y) = \inf_{\lambda \in \Lambda} \left\{ \max \left(\gamma(\lambda), \int_0^\infty e^{-s} \rho_s(x, y, \lambda) ds \right) \right\}.$$

a) Show that ρ is a metric on $D_Y[0, \infty)$ (sometimes called the Skorohod metric and its topology is called the Skorohod topology on $D_Y[0, \infty)$).

b) Let (x_n) be a sequence in $D_Y[0, \infty)$ and $x \in D_Y[0, \infty)$. Show that $\rho(x_n, x) \to 0$ if and only if there is a sequence (λ_n) in Λ so that $\gamma(\lambda_n) \to 0$ and $\rho_s(x_n, x, \lambda_n) \to 0$ at all points s where x is continuous.

Hint: In b), you'll probably need to use the result that each $x \in D_Y[0, \infty)$ has only countably many points of discontinuity.

4. Let X be a continuous time Markov chain on a countable space S. We can interpret X as a mapping $X : \Omega \to D_S[0, \infty)$, where Ω is the product of the sample space of a discrete time Markov chain and the sample space of a sequence of i.i.d. random times with exponential distribution and mean one. Let Σ be the product σ-algebra on Ω (the product of the discrete time Markov chain σ-algebra and the σ-algebra of the i.i.d. exponential times) and $\mathcal{F}$ be the σ-algebra generated by the cylinder sets of $D_S[0, \infty)$. Show that $X : (\Omega, \Sigma) \to (D_S[0, \infty), \mathcal{F})$ is measurable.

5. Let S be a countable abelian group (stick to $S = \mathbb{Z}^d$ if you want to) and $p_t(x, y)$ be a transition probability which is symmetric and translation invariant: $p_t(x, y) = p_t(y, x) = p_t(0, y - x)$. Let X and Y be independent identically distributed continuous time Markov cahins on S with transition probabilities p_t. Show that $Z = X - Y$ has the same distribution of $(X_{2t})_{t \geq 0}$, i.e. the process run at double speed.

Hint: It is enough to check that the finite dimensional distributions agree, i.e. you'll probably want to show that

$$P^{x,y}(Z_{t_1} = z_1, ..., Z_{t_n} = z_n)$$

factors nicely like a Markov chain should, only depends on $x - y$ and agrees with the corresponding quantity for $(X_{2t})_t$. To check that it really is enough that finite dimensional distributions agree, you'll need a monotone class argument (or sometimes called Dynkin's π-λ theorem) or just look it up in any good book on probability.