STOCHASTIC PARTICLE SYSTEMS - EXERCISE 1

Some starting remarks:

N.B. Due to Ferrari's course, the first exercise session is exceptionally on THURSDAY the 10th at 12.15. The classroom is B120. Usually the exercises will be on fridays at 14.15 in the classroom C122.

If you have any questions related to the exercises or the course in general, feel free to contact the assistant (room B418, e-mail: christian.webb@helsinki.fi).

If you can't attend the exercise session, you can hand in (or e-mail) your solutions to the assistant.

When dealing with measures on countable sets, we shall usually assume that every set is measurable so that it is enough to define the measure on singletons and extend it to other sets via countable additivity.

1. For $i \in \{1, ..., N\}$, let X_i be a Poisson distributed random variable with parameter λ_i independent of X_j for $j \neq i$. Show that $\sum_{i=1}^N X_i$ is Poisson distributed with parameter $\sum_{i=1}^N \lambda_i$.

2. *The setup:* Consider the set $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d$, i.e. the discrete d-dimensional torus and let Ω be the set of occupation configurations on $\mathbb{T}_N^d$, i.e. $\eta \in \Omega$ is a map $\eta : \mathbb{T}_N^d \to \mathbb{N} = \{0, 1, 2, ...\}$. The interpretation is that for any $x \in \mathbb{T}_N^d$, $\eta(x)$ tells you how many particles are located at x. Consider now the probability measure μ_α on Ω defined by

$$\mu_\alpha(\eta) = \prod_{x \in \mathbb{T}_N^d} \frac{\alpha^{\eta(x)} e^{-\alpha}}{\eta(x)!}.$$

What this means is that the number of particles at each x is independent of the number of particles at the other points and it is Poisson distributed with parameter α (which is constant in x). The total number of particles is now a random variable under this distribution (it is Poisson distributed according to problem 1). To relate this measure to the random walk scenario, we condition on the event that the total number of particles is K: let $S_K = \{\eta \in \Omega : \sum_x \eta(x) = K\}$ and let us define the measure $\tilde{\mu}$ on Ω by

$$\tilde{\mu}(\eta) = \frac{\mu_\alpha(\eta; \eta \in S_K)}{\mu_\alpha(S_K)}.$$

The actual problems:

a) Show that $\tilde{\mu}$ is independent of α.

b) Show that if we take K independent simple random walks $\{X^i : i = 1, ..., K\}$ on $\mathbb{T}_N^d$ and let each of their initial distributions be uniform, then $\tilde{\mu}$ is the distribution of $x \mapsto \sum_{i=1}^K \mathbf{1}_{\{X^i = x\}}$

3. Let (Y, ρ) be a metric space, $\mathcal{B}(Y)$ be the σ-algebra of the Borel sets of Y and $\mathcal{M}_1(Y)$ the set of probability measures on $(Y, \mathcal{B}(Y))$. For $A \subset Y$ and $\epsilon > 0$, let

$$A^{(\epsilon)} = \{x \in Y : \rho(x, y) < \epsilon \text{ for some } y \in Y\}.$$

Define

$$r(\mu, \nu) = \inf\{\epsilon > 0 : \nu(F) \leq \mu(F^{(\epsilon)}) + \epsilon \text{ for every closed set } F \subset Y\}.$$

Show that r satisfies the triangle inequality, i.e. that for each $\lambda, \mu, \nu \in \mathcal{M}_1(Y)$, $r(\lambda, \nu) \leq r(\lambda, \mu) + r(\mu, \nu)$.

4. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra and $X : \Omega \to \mathbb{R}$ some random variable. Recall that the conditional expectation of X given $\mathcal{G}$ defined to be any $\mathcal{G}$ measurable random variable $\tilde{X}$ satisfying

$$\int_G \tilde{X}(\omega) dP(\omega) = \int_G X(\omega) dP(\omega)$$

for all $G \in \mathcal{G}$. Moreover, the conditional expectation is almost surely defined uniquely by this condition and one usually writes $\tilde{X} = E(X|\mathcal{G})$. If $Y : \Omega \to \mathbb{R}$ is a $\mathcal{F}$-measurable mapping one can consider the σ-algebra generated by Y: $\sigma(Y) \subset \mathcal{F}$. This is the smallest σ-algebra we can put on Ω that makes Y a measurable map. In this case, one often writes $E(X|\sigma(Y)) = E(X|Y)$.

Consider now the special case that Y takes only finitely many values. Show that there is some function $f : \mathbb{R} \to \mathbb{R}$ so that $E(X|Y) = f(Y)$, i.e. that for every $\omega \in \Omega$, $E(X|Y)(\omega) = f(Y(\omega))$.

5. Let S_n be a simple random walk on $\mathbb{Z}$. Show that $X_n = \max\{S_m : 0 \leq m \leq n\}$ is not a Markov chain.