We get, since $S(t)$ contracts in L^2:

$$0 \leq \hat{S}(t,k) \leq \lim_{\Lambda} (f_\Lambda, f_\Lambda) = \hat{S}(0,k)$$

Spectral theorem gives

$$(f_{\Lambda,k}, S(t) f_{\Lambda,k}) = \int_{\mathbb{R}^+} d\nu(\lambda)\, e^{-t\lambda}$$

where

$$(f_{\Lambda,k}, L f_{\Lambda,k}) = \int \lambda \, d\nu_\Lambda(\lambda)$$

Thus

$$(f_{\Lambda,k}, S(t) f_{\Lambda,k}) = S(0,k) \underbrace{\frac{\int d\nu_\Lambda(\lambda)\, e^{-t\lambda}}{\int d\nu_\Lambda(\lambda)}}_{(*)}$$

Jensen's inequality $\Rightarrow$

$$(*) \geq \exp - t \frac{\int \lambda \, d\nu_\Lambda}{\int d\nu_\Lambda} = \exp\left(-t \frac{(f_{\Lambda,k}, L f_{\Lambda,k})}{(f_{\Lambda,k}, f_{\Lambda,k})}\right)$$

Now $\lim (f_{\Lambda,k}, L f_{\Lambda,k}) = \sum_x e^{ikx} \underbrace{E^\mu (f_0 \, L f_x)}_{= g(x)}$

where $f_x(\eta) = \eta(x) - \rho$. $g(x) = (f_0, L f_x) =$

$= (L f_0, f_x)$ by reversibility.

Also $g(x) = (f_{-x}, L f_0)$ by translation invariance

Hence got $g(x) = \overline{g(-x)}$.

Since $L\rho = 0$, $g(x) = (\eta(0), L\eta)(x))$

We have $\sum_x g(x) = 0$. by conservation of particle number. Indeed,

$$L\eta(x) = \frac{1}{2}\sum_{u,v} c(u,v,\eta)\,(\underbrace{\eta^{uv}(x)-\eta(x)}_{=0 \text{ unless } x=u \text{ or } x=v})$$

$$= \frac{1}{2}\sum_{v} c(x,v,\eta)\,(\eta(v)-\eta(x))$$

$$+\frac{1}{2}\sum_{u} c(u,x,\eta)\,(\eta(u)-\eta(x))$$

(finite sums, $|x-v|<R$, $|x-u|<R$

Thus ($g(x)$ decays exponentially or

absolutely summable)

$$\sum_x g(x) = \sum_{x,u}(\eta(0),c(x,u,\eta)+c(u,x,\eta))\,(\eta(u)-\eta(x))$$

$$= 0$$

We got, after $\Lambda\to\infty$:

$$\hat{S}(0,k) \geq \hat{S}(t,k) \geq \hat{S}(0,k)\exp\left[-t\,\frac{A(k)}{\hat{S}(0,k)}\right]$$

$$A(k) = \frac{1}{2}\sum_x (1-\cos kx)\,g(x)$$

$$g(x) = (\eta(0), L\eta(x)) = \sum_y (\eta(0), c(x,y,\eta)(\eta(y)-\eta(x))$$

We have $A(k) = \frac{1}{4}\sum_{ij} a_{ij}k_ik_j + o(k^2)$ as $k\to 0$

$$a_{ij} = \sum_x x_i x_j\, g(x)$$

Also

$$\int_{[-\pi,\pi]^d} \frac{dk}{(2\pi)^d}\, \hat{S}(\ell,k) = E^\mu(\eta_\ell(0)\,\eta_0(0)) - \rho^2$$

$$\hat{S}(0,0) = \sum_x E^\mu(\eta(x)\eta(0)) - \rho^2 \equiv \chi$$

(since $E^\mu(\eta(0)\eta(0)) = E^\mu(\eta(0)) = \rho$, $\hat{S}(0,0) \geq \rho-\rho^2 > 0$)

χ is "static compressibility".

Hence

$$E^{\mu}(\eta_t(0)\eta_0(0)) - \rho^2 \geq \int \frac{dk}{(2\pi)^d} \hat{S}(0,k)\, e^{-t\frac{A(k)}{\hat{S}(0,k)}}$$

Exercise Compute a_{ij} for symmetric exclusion $p(x,y) = \begin{cases} 1 & |x-y|=1 \\ 0 & \text{other} \end{cases}$

and show $a > 0$ (as a matrix)

Typically: $a > 0$ so $\frac{A(k)}{\hat{S}(0,k)} \underset{k\to 0}{\sim} k^2$

get

$$E^{\mu}(\eta_t(0)\eta_0(x)) - \rho^2 \underset{t\to\infty}{\geq} \text{const}\ \frac{1}{t^{d/2}}$$

i.e. correlations in time decay only as a power law (not exponentially).

The lower bound

$$\hat{S}(t,k) \geq c\, e^{-tDk^2}$$

is consistent with

$$S(t,x) \leq c\, t^{-d/2} e^{-x^2/4Dt}$$

which is diffusive behaviour.

A challenging conjecture is:

$$\lim_{L\to\infty} L^d S(L^2 t, Lx) = \chi \frac{1}{\sqrt{\det D}} \frac{1}{(4\pi t)^{d/2}} e^{-\frac{(x, D^{-1}x)}{4t}}$$

D positive matrix

Green - Kubo formula

Define, whenever exists, the diffusion matrix

$$D_{\alpha\beta} = \lim_{t\to\infty} \frac{1}{t} \cdot \frac{1}{\chi} \sum_{x\in\mathbb{Z}^d} x_\alpha x_\beta \, S(t, x)$$

where

$$\chi = \sum_x S(0, x)$$

We will show the limit exists and we will

relate $D_{\alpha\beta}$ to a current correlation function.

We have $Lf(\eta) = \frac{1}{2} \sum_{x,y} c(x,y,\eta)\,(f(\eta^{xy}) - f(\eta))$

$\eta_t(x)$ changes in time due to jumps of

particles to x or from x.

Rate of jumps $x \to y$ $= c(x,y,\eta)\,\eta(x)(1-\eta(y))$

— " — $y \to x$ $= c(y,x,\eta)\,\eta(y)(1-\eta(x))$

Flux of particles from $x \to y$:= $j(x,y,\eta)$ =

$c(x,y,\eta)[\eta(x)(1-\eta(y)) - \eta(y)(1-\eta(x))] = c(x,y,\eta)\,(\eta(x)-\eta(y))$

Let $\eta_\cdot \in D[0,\infty]$ be a trajectory of our process

$t \to \eta_t(x) \in \{0,1\}$ consists of jumps

Let $J_{xy}([s,t])$ = # jumps $x \to y$ during time $[s,t]$ − # jumps $y \to x$

We can define it as a process coupled to

η as follows.

Define Markov process (η_t, N_t) ; $\eta_t \in \{0,1\}^{\mathbb{Z}^d}$, $N_t \in \{0,1\}$ by generator

$$Lf(\eta, N) = \frac{1}{2} \sum_{(x',y') \neq (x,y)} c(x',y',\eta)\,(f(\eta^{x'y'}, N) - f(\eta, N))$$

$$+\, c(x,y,\eta)\,\eta(x)(1-\eta(y))\,[f(\eta^{xy}, N+1) - f(\eta, N)]$$

$$+\, c(x,y,\eta)\,(1-\eta(x))\,\eta(y)\,[f(\eta^{xy}, N-1) - f(\eta, N)]$$

We set $J_{xy}[s,t] = N_t - N_s$.

Recall that for $f \in \mathcal{D}(L)$

$$f(\eta_t, N_t) - \int_0^t (Lf)(\eta_s, N_s)\,ds$$

is a Martingale. Take $f(\eta, N) \equiv N$ and $N_0 = 0$. Then $Lf = c(x,y,\eta)[\eta(x)(1-\eta(y)) - \eta(y)(1-\eta(x))] = j(x,y,\eta)$ so

$$M_{xy}(t) = J_{xy}([0,t]) - \int_0^t j(x,y,\eta_{t'})\,dt'$$

is a Martingale.

<u>Prop</u> Let μ be stationary measure. Then

$$E^\mu M_{xy}(t) M_{x'y'}(t) = t(\delta_{xx'}\delta_{yy'} - \delta_{xy'}\delta_{yx'}) \cdot E^\mu(c(x,y,\eta))$$

<u>Pf</u> We need a Lemma

<u>Lemma</u> Let $X_t \in Y$ be a Markov process with generator L. Let $F \in \mathcal{D}(L)$ be s.t. $F^2 \in \mathcal{D}(L)$. Then

$$M(t) := F(X_t) - \int_0^t LF(X_s)\,ds$$

$$K(t) := M(t)^2 - \int_0^t \left((LF)^2(X_s) - 2F(X_s)LF(X_s)\right)ds$$

are Martingales.

Assuming the Lemma, let us define

$N(t) = J_{xy}([0,t])$ and $N'(t) = J_{x'y'}([0,t])$

as above (add to the generator above also x', y' variables i.e. $S(\eta, N, N')$).

Apply lemma to F_1, F_2, F_1+F_2. Get

$$M_1(t)\,M_2(t) - \int_0^t \left(L(F_1F_2) - F_1LF_2 - F_2LF_1\right)ds$$

is Martingale. So

$$0 = E\left(M(t)M'(t)\right) - \int_0^t L(NN') - NLN' - N'LN$$

So by stationarity, if $\{x,y\} \neq \{x',y'\}$,

$$E\,M(t)M'(t) = t\,E^{\mu}\left(L(NN') - NLN' - N'LN\right)$$

We compute $L(NN') - (LN\,N' + NLN')$ $\Rightarrow$ get $= 0$.

$$E\,M^2 = t\,E\left(LN^2 - 2NLN\right) = t\,E(x,y,\eta)\left[\eta(x)(1-\eta(y))\right.$$
$$\left.\left((N+1)^2 - N^2\right) + 2N(N+1-N)\right) + \eta(y)(1-\eta(x))\left((N-1)^2 - N^2\right) + 2N(N-1-N)\left.\right]\Big]$$

$$= \ell \; E\, c(x,y,\eta)\,\big(\eta(x)(1-\eta(y)) + \eta(y)(1-\eta(x))\big)$$

Let us define $c(x,y,\eta) = 0$ if $\eta(x) = \eta(y)$

(this is no loss!). Since $(\ldots) = 1$ if $\eta(x) \neq \eta(y)$

get $= \ell \; E\, c(x,y,\eta)$ $\square$

We can now compute the current-current correlation function:

<u>Exercise</u> Show $E\left(J_{xy}[0,t] - \int_0^t j(x,y,\eta_s)\,ds\right) = 0$

using reversibility

Then

$$E^{\mu}\left(J_{xy}[0,t]\; J_{x'y'}[0,t]\right) = t\left(\delta_{xx'}\delta_{yy'} - \delta_{xy'}\delta_{yx'}\right)$$

$$\cdot\; E^{\mu}\left(c(x,y,\eta)\right) - \int_0^t ds \int_0^t ds'\; E^{\mu}\left(j(x,y,\eta_s)\; j(x',y',\eta_{s'})\right)$$

Define

$$j_\alpha(\eta) = \frac{1}{2} \sum_y y_\alpha\, j(0,y,\eta)$$

Then we have

<u>Prop.</u> The limit

$$D_{\alpha\beta} = \lim_{t\to\infty} \frac{1}{t}\,\frac{1}{2\chi} \sum_x x_\alpha x_\beta\, S(x,t)$$

exists and is given by

$$D_{\alpha\beta} = \frac{1}{2\chi}\Big(\frac{1}{2}\delta_{\alpha\beta} \sum_x x_\alpha^2\, E^{\mu}\left(c(0,x,\eta)\right)$$

$$- 2\int_0^\infty dt \sum_x E^{\mu}\left(j_\alpha\, S(t)\, \tau_x j_\beta\right)\Big)$$

where the sums and integrals converge

Pf. We have from conservation of particles

$$\eta_t(x) - \eta_0(x) = - \sum_y J_{xy}[0,t]$$

so

$$\sum_x x_\alpha x_\beta \left(S(t,x) - S(0,x)\right) = -\tfrac{1}{2} \sum x_\alpha x_\beta E\left((\eta_t(x) - \eta_0(x))(\eta_t(0) - \eta_0(0)) \right)$$

$$= -\tfrac{1}{2} \sum_x \sum_{yy'} x_\alpha x_\beta \; E\left(J_{xy}[0,t] \, J_{0y'}[0,t] \right)$$

$$= \frac{t}{2} \sum_x x_\alpha x_\beta \, E\left(c(x,0,\eta)\right) - \underbrace{\frac{1}{2} \sum_{xyy'} \int_0^t ds \int_0^t ds'}_{*}$$

$$x_\alpha x_\beta E\left(j(x,y,\eta(s)) \, j(0,y',\eta(s')) \right)$$

Recalling $j(x,y,\eta) = c(x,y,\eta)(\eta(x) - \eta(y)) = -j(y,x,\eta)$

some algebra gives (exercise)

$$* = -2 \int_0^t ds \int_0^t ds' \sum_x E\left(j_\alpha(\eta(s)) \tau_x j_\beta(\eta(s')) \right)$$

Now

$$E\left(j_\alpha(\eta(s)) \tau_x j_\beta(\eta(s')) \right) = \left(j_\alpha, S(|s-s'|) \tau_x j_\beta \right)$$

$$\equiv G_{\alpha\beta}(|s-s'|, x) \quad = \text{symmetric matrix.}$$

Let $v \in \mathbb{R}^d$, $G_v = \sum v_\alpha v_\beta G_{\alpha\beta}$, $j = \sum j_\alpha v_\alpha$

$$G_v(T,x) = \left(j, S(T) \tau_x j \right)$$

This decays exponentially in x for fixed T (proof as for $S(t,x)$, use $Ej = 0$)

Thus we may write

$$\sum_x G_\nu(T,x) = \lim_{\Lambda \nearrow \mathbb{Z}^d} (j_\Lambda, S(T)\, j_\Lambda)$$

$$j_\Lambda \equiv \frac{1}{\sqrt{\Lambda}} \sum_{x\in\Lambda} j(x) .$$

We have obtained

$$\frac{1}{t} \sum_x (v\cdot x)^2 \, S(t,x) = \frac{1}{t} \sum_x (v\cdot x)^2 S(0,x)$$

$$+ \frac{1}{2} \sum_x (v\cdot x)^2 \, E\,(\, c(0,x,\eta)\,) - \frac{1}{t}\int_0^t ds \int_0^s ds' \lim_{\Lambda\nearrow\infty} (j_\Lambda, S(s-s'))\, j_\Lambda)$$

<u>Lemma</u> LHS ≥ 0.

<u>Pf</u>

$$\text{LHS} = -\frac{1}{t} \sum_{\alpha\beta} v_\alpha v_\beta \, \frac{\partial^2 \hat{S}(t,k)}{\partial k_\alpha \partial k_\beta}\Big|_{k=0}$$

But

$$\hat{S}(t,k) = \lim_{\Lambda\nearrow\mathbb{Z}^d} (\,\xi_{\Lambda,k}\,, S(t)\,\xi_{\Lambda,k}) - \rho^2$$

$$\xi_{\Lambda,k} = \frac{1}{\sqrt{\Lambda}} \sum_{y\in\Lambda} e^{-iky}\, \eta(y) \qquad (\text{for } k\neq 0)$$

$$|\xi_{\Lambda,k}| \leq \xi_{\Lambda,0} \;\Rightarrow\; \hat{S}(t,k) \leq \hat{S}(t,0)$$

Also $\partial_k \hat{S}(t,k)\big|_{k=0} = 0$ so $k=0$ is a maximum $\Rightarrow$ claim $\square$