Convergence to equilibrium

Suppose μ is invariant: $S(t)^*\mu = \mu$ $\forall t$.

Given $\mu_0 \in \mathcal{M}_1(X)$ when do we have

$$S(t)^*\mu_0 \underset{t\to\infty}{\Rightarrow} \mu \quad \text{(weakly)} \quad ?$$

Let us consider $\mu_0 \ll \mu$ i.e.

$$\mu_0 = f\mu \qquad f \in L^1(\mu) \quad , f \geq 0$$

i.e. $\int f \, d\mu = 1$.

<u>Lemma</u> a) Let μ be invariant. Then

$$S(t): L^1(\mu) \to L^1(\mu) \quad \text{is contraction}$$

$$\| S(t) f \|_1 \leq \| f \|_1 \equiv \int |f| \, d\mu$$

b) Let μ be reversible. Then

$$S(t) : L^2(\mu) \to L^2(\mu)$$

is strongly continuous contraction semigroup

$$\| S(t) f \|_2 \leq \| f \|_2$$

and $S(t)$ is self adjoint $(f, S(t)g) = (S(t)f, g)$

The generator L is self adjoint with domain the closure of $\mathcal{D}(L) \subset C(X)$ in L^2

<u>Pf.</u> a) $\| S(t) f \|_1 = \int d\mu(\eta) | E^\eta(f(\eta_t)) |$

$$\leq \int d\mu(\eta) \, E^\eta(|f(\eta_t)|) = \int S(t)|f| \, d\mu = \int |f| \, d\mu = \| f \|_1$$

b) Let $f \in C(X)$.

$$[S(t)f(\eta)]^2 = (E^\eta f(\eta_t))^2$$

$$\underset{\text{Schwartz}}{\leq} \quad E^\eta f^2 = (S(t)f^2)(\eta)$$

So $\int (S(t)f)^2 \, d\mu \leq \int S(t)f^2 = \int f^2 \, d\mu$

$C(X)$ dense $\Rightarrow \| \tilde{S(t)} \| \leq 1$ in L^2.

So $S(t)$ is a continuous semigroup in L^2.

Generator:

$$\mathcal{D}(\tilde{L}) = \{ f \in L^2 \mid \lim \frac{S(t)f - f}{t} = g \in L^2 \}$$

Since $\mathcal{D}(L) \subset C(X) \subset L^2$ get $\mathcal{D}(\tilde{L}) \supset \mathcal{D}(L)$

so $\tilde{L}$ is an extension of L.

We claim $\tilde{L}$ is the closure of L i.e.

$$\text{Graph}(\tilde{L}) = \overline{\text{Graph}(L)} \leftarrow \text{in } L^2.$$

Since $\tilde{L}$ is extension and closed $\text{Graph}\,\tilde{L} \supset \overline{\text{graph}(L)}$

Let $(f, g) \in \overline{\text{graph}(L)}$. $\exists f_n \in \mathcal{D}(L)$ s.t.

$f_n \to f$ (in L^2) $\quad Lf_n \to g$ in L^2

But $Lf_n = \tilde{L} f_n$ $\quad (\mathcal{D}(L) \subset \mathcal{D}(\tilde{L}))$ so $\tilde{L} f_n \to g$, $\quad f_n \to f$

$\Rightarrow g = \tilde{L} f$ so $(f, g) \in \text{graph}\,\tilde{L}$ $\square$ $\quad \Downarrow \mathcal{D}(\tilde{L})$

Recall: $(f, Lg) = (Lf, g) \ \forall f, g \in \mathcal{D}(L) \Rightarrow$
$(f, \tilde{L}g) = (\tilde{L}f, g) \ \forall f, g \in \mathcal{D}(\tilde{L})$. So $\tilde{L}$ symmetric
$\Rightarrow$ Adjoint of $\tilde{L}$ extends $\tilde{L}$. Adjoint of $\tilde{L}$ is
also Markov generator (Check conditions!). But then $\tilde{L} = \tilde{L}^*$
$\square$

Now we have

$$S(t)^*(f\mu) = (S(A)f)\mu$$

<u>Proof</u> $(h, S(t)f) = (S(t)h, f)$ so

$$\int h\,(S(t)f)\,d\mu = \int S(t)h\, f\, d\mu$$

$$= \int h \; d(S(t)^* f\, d\mu) \qquad \forall h \in C(X)$$

so claim $\square$

Hence, to show $S(t)^*(f\mu) \to \mu \iff$

show $\underbrace{(h, (S(t)f - 1))}_{(*)} \to 0 \quad \forall h \in C(X)$

$$(*) \leq \|h\|_\infty \int |S(t)f - 1|\, d\mu$$

so suffices to show

$$\int |S(t)f - 1|\, d\mu \underset{t\to\infty}{\longrightarrow} 0$$

Suppose first $f \in L^2$. L is self-adjoint and $\mathrm{Range}(\lambda - L) = L^2$ for $\lambda > 0$ so L is negative: $\mathrm{spectrum}(L) \subset (-\infty, 0]$.

Spectral theorem $\Rightarrow$

$$-\tilde{L} = \int_0^\infty \lambda\, dP_\lambda \quad , \quad S(t) = \int_0^\infty e^{-\lambda t}\, dP_\lambda$$

so $S(t)f \underset{t\to\infty}{\longrightarrow} P_0 f$, P_0 orthogonal projecti to

$$\mathcal{J} = \{ g \in L^2 \mid S(t) g = g \quad \forall t > 0 \}$$

Now, $\mathcal{J} \subset \mathcal{D}(\tilde{L})$ since $\dfrac{S(t)g - g}{t} = 0 \quad \forall t,\ g \in \mathcal{J}$.

<u>Lemma</u> Let $f, g \in \mathcal{D}(\tilde{L})$. Then

$$-(f, \tilde{L} g) = \frac{1}{2} \sum_{x,y} \int c(x,y,\eta) \overline{(f(\eta^{xy}) - f(\eta))} (g(\eta^{xy}) - g(\eta)) \, d\mu$$

and series converges absolutely.

<u>Pf</u> Let first $f, g \in \mathcal{C}$. Then series has finite # of terms.

$$(f, \tilde{L} g) = \sum_{x,y} \int \overline{f(\eta)} (g(\eta^{xy}) - g(\eta)) \, c(x,y,\eta) \, d\mu$$

Recall detailed balance: $c(x,y,\eta) = c(x,y,\eta^{xy}) e^{-\Delta_{xy} H}$

This implies

$$= \sum_{xy} \int \overline{f(\eta^{xy})} (g(\eta) - g(\eta^{xy})) \, c(x,y,\eta) \, d\mu$$

by $e^{-\Delta_{xy} H} d\mu(\eta) = d\mu(\eta^{xy})$. So

$$(f, \tilde{L} g) = \frac{1}{2} \sum_{xy} \int \overline{(f(\eta^{xy}) - f(\eta))} (g(\eta^{xy}) - g(\eta)) \, c(x,y,\eta) \, d\mu$$

Claim follows by limits (exercise). $\square$

Thus $g \in \mathcal{J} \Rightarrow$

$$\sum_{xy} \int |g(\eta^{xy}) - g(\eta)|^2 c(x,y,\eta) \, d\mu = 0$$

Assume $c(x,y,\eta) > 0 \quad \forall x, y,\ |x-y| = 1$. Then

$$g(\eta^{xy}) = g(\eta) \quad \forall x, y, \quad \text{a.e. in } \mu.$$

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If μ is a high temperature Gibbs state (eg. Bernoulli), then this $\Rightarrow$ $g(\eta) = $ constant a.e. in μ. Note also that $1 \in \mathcal{J}$.

Hence we get, for $f \in L^2$

$$S(t)f \to 1 \quad \text{in } L^2(\mu) \qquad \left(P_0 f = (1,f) = \int f d\mu = 1\right)$$

To get $f \in L^1$, pick $f_n \in L^2$, $f_n \to f$ in L^1 and $f - f_n \geq 0$ (eg $f_n = f \cdot \mathbb{1}_{|f|<n}$)

Then

$$\int |S(t)f - 1| \, d\mu = \overset{(a)}{\int |S(t)(f - f_n)| \, d\mu} + \underset{(A)}{\int |S(t)f_n - 1| \, d\mu}$$

$$(A) \leq \left(\int |S(t)f_n - 1|^2 d\mu\right)^{1/2} \xrightarrow[t\to\infty]{} 0$$

(a) : $f - f_n \geq 0 \Rightarrow S(t)(f - f_n) \geq 0 \Rightarrow$

$$(a) = \int S(t)(f - f_n) \, d\mu = \int (f - f_n) \, d\mu$$

$$= \int |f - f_n| \, d\mu \xrightarrow[n\to\infty]{} 0$$

We got

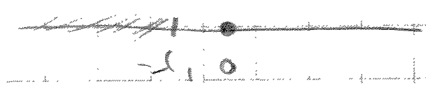
<u>Prop</u> If Range $(P_0) = \{\text{constants}\}$

$$S(t)f \to \int f d\mu \qquad \forall f \in L^1(\mu)$$

Speed of convergence?

Spectrum of L ($\equiv \sigma(L)$) lies in $(-\infty, 0]$. 0 is simple eigenvalue. Define *spectral gap*

$$\lambda_1 = \inf_{\substack{f \in L^2 \\ (f,1)=0}} \frac{-(f, Lf)}{(f,f)}$$

Suppose $\lambda_1 > 0$. Then $\sigma(L) =$ 

so
$$S(t)f = P_0 f + \int_{\lambda_1}^{\infty} e^{-\lambda t}\, dP_\lambda f$$

$P_0 f = (1,f)$. Thus

$$\| S(t)f - P_0 f \|_2^2 = (f - P_0 f,\ S(2t) f - P_0 f)$$
$$= \int_{\lambda_1}^{\infty} e^{-2\lambda t}\, \mu(d\lambda) \le e^{-2\lambda_1 t} \int \mu(d\lambda)$$
$$= e^{-2\lambda_1 t} \| f - P_0 f \|^2$$

So exponential convergence.

In our case $\lambda_1 = 0$!

If consider *finite volume* $\eta \in \{0,1\}^{\mathbb{T}_N^d}$

then $\lambda_1 \propto 1/N^2 \xrightarrow[N \to \infty]{} 0$.

This is much deeper!

Time correlations in equilibrium

Let μ be transl. invariant invariant measure

- Gibbs measure

or

- Bernulli for exclusion process

Let

$$\rho = \int \eta(x)\, \mu(d\eta) = E^{\mu}(\eta_t(x)) \quad \forall t$$

be the density.

Correllation function $t > s$

$$E^{\mu} \eta_t(x) \eta_s(y) = E^{\mu} \eta_{t-s}(x-y) \eta_0(0)$$

So define

$$S(t,x) = E^{\mu} \eta_t(x) \eta_0(0) - E\eta_t(x) E\eta_0(0)$$

$$= E^{\mu} \eta_t(x) \eta_0(0) - \rho^2$$

We get

$$|S(t,x)| = |E(\eta_t(x)-\rho)(\eta_0(0)-\rho)|$$

$$\leq \left(E(\eta_t(x)-\rho)^2\, E(\eta_0(0)-\rho)^2 \right)^{1/2}$$

Schwartz

$$= \int \underbrace{(\eta(x)-\rho)^2}_{\substack{= \eta(x) - 2\rho\eta(x) + \rho^2 \\ \uparrow \\ \eta^2 = 1}} d\mu = \rho(1-\rho)$$

<u>Prop</u> $|S(t,x)| \le c_1(t)\, e^{-c_2(t)|x|}$ $c_1(t) < \infty$, $c_2(t) > 0$

i.e. for <u>fixed time</u> correllations decay exponentially.

Let us do some preliminaries. (We'll also give a construction of $S(t)$ on the way!)

We suppose $c(x,y,\eta) = 0$ if $|x-y| \ge R$. Let Λ be cube, center origin.

<u>Let</u> $\mathcal{C}_\Lambda = \{ f \in C(X) \mid f \text{ local in } \Lambda : f(\eta) = f(\eta|_\Lambda) \}$

suppose also $c(x,y,\eta) \in \mathcal{C}_{B_R(x)}$ $B_R(x) = \{ y \mid |x-y| \le R \}$

$$L_\Lambda f := \frac{1}{2} \sum_{\substack{x \in \Lambda \\ y \in \mathbb{Z}^d}} c(x,y,\eta)\,(f(\eta^{xy}) - f(\eta)) \qquad |\Lambda| < \infty, \ \Lambda \subset \mathbb{Z}^d$$

Note: only $y \in \bar{\Lambda} = \{ y \mid \operatorname{dist}(y,\Lambda) \le R \}$ occur.

L_Λ generates Markov process in the finite state space $\{0,1\}^{\bar{\Lambda}}$, semigroup $S_\Lambda(t)$.

<u>Lemma</u> Let $f \in \mathcal{C}_\Lambda$. Then for all $\Lambda' \subset \mathbb{Z}^d$ and all n

$$\| (L_{\Lambda'})^n f \| \le n!\, e^{|\Lambda|}\, C(R)^n \, \|f\|$$

This holds for $\Lambda' = \mathbb{Z}^d$ i.e. for L too.

<u>Pf</u> Let $\mathcal{L}_x f = \sum_y c(x,y,\eta)\,(f(\eta^{xy}) - f(\eta))$

so that $L_{\Lambda'} = \sum_{x \in \Lambda'} \mathcal{L}_x$.

We have

$$L^n_{\Lambda'} f = \sum_{x_1,\dots,x_n \in \Lambda'} \delta_{x_n} \delta_{x_{n-1}} \cdots \delta_{x_1} f$$

We have

$\delta_{x_1} f = 0$ unless $x_1 \in \overline{\Lambda}$. Moreover $\delta_{x_1} f \in \mathcal{C}_{\Lambda(x_1)}$

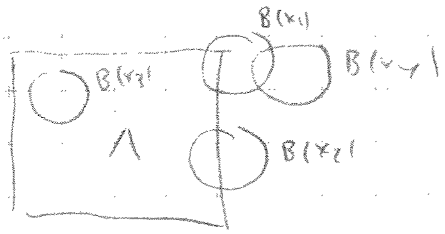
$$\Lambda(x_1) = \Lambda \cup B_R(x_1)$$

Hence $\delta_{x_2} \delta_{x_1} f = 0$ unless $x_2 \in \overline{\Lambda(x_1)}$ ~~so~~

$\delta_{x_2} \delta_{x_1} f \in \mathcal{C}_{\Lambda(x_1,x_2)}$, $\Lambda(x_1,x_2) = \Lambda(x_1) \cup B_R(x_1)$

So (lets consider $\Lambda' = \mathbb{Z}^d$ for simplicity)

$$L^n f = \sum_{x_1 \in \overline{\Lambda}} \sum_{x_2 \in \overline{\Lambda(x_1)}} \cdots \sum_{x_n \in \overline{\Lambda(x_1,\dots,x_{n-1})}} \delta_{x_n} \cdots \delta_{x_1} f$$



We have : $\| \delta_x \| \leq 2 \sup_z \sum_y c(0,y,z) \equiv \lambda$

Thus

$$|\Lambda(x_1,\dots x_k)| \leq C(|\Lambda| + kR^d)$$

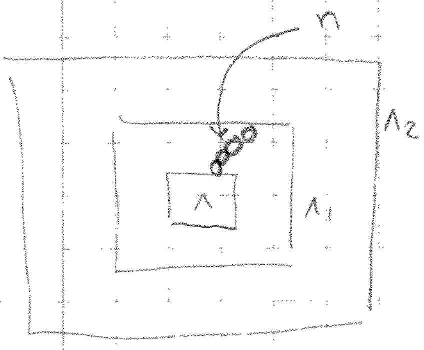
so

$$\|L^n f\| \leq (C\lambda)^n \prod_{k=1}^{n} (|\Lambda| + kR^d) \|f\|$$

$$\leq (C\lambda)^n n! \frac{(|\Lambda| + nR^d)^n}{n!} \|f\|$$

$$\leq (C\lambda)^n n! \exp(|\Lambda| + nR^d) \|f\| \quad \square$$

Now, let $f \in \mathcal{C}_\Lambda$ and let $\Lambda \subset \Lambda_1 \subset \Lambda_2$



Then $L^n_{\Lambda_1} f = L^n_{\Lambda_2} f$

for $n \leq c \, dist(\Lambda, \Lambda_1^c) \equiv n(\Lambda, \Lambda_1)$

$(c \sim 1/R)$.

We get from Lemma

$$\| S_{\Lambda_1}(t) f - S_{\Lambda_2}(t) f \| \leq 2 \|f\| e^{|\Lambda|} \underbrace{\sum_{n=n(\Lambda,\Lambda_1)}^{\infty} (ct)^n}_{(*)}$$

Thus for $t \leq t_0$ $(*) \to 0$ as $\Lambda_1 \uparrow \mathbb{Z}^d$.

So

$$S_{\Lambda_1}(t) f \to g \in C(X) \qquad (*)$$

uniformly in $t \leq t_0$. Denote $g = S(t) f$.

Lemma gives

$$S(t) f = \sum \frac{t^n}{n!} L^n f$$

Since $S_{\Lambda_1}(t)$ is contractive semigroup get $\|S(t) f\| \leq \|f\|$, from $(*)$, for all $f \in \mathcal{C}$. $\mathcal{C}$ is dense $\Rightarrow$ $S(t)$ extends to all of $C(X)$. Also $(*)$ extends to all of $C(X)$:

$$\| S_\Lambda(t) f - S(t) f \| \xrightarrow[\Lambda \nearrow \mathbb{Z}^d]{} 0 \qquad \forall f \in C(X)$$

Since $S_\Lambda(t_1 + t_2) = S_\Lambda(t_1) S_\Lambda(t_2)$ get the same for S if $t_1 + t_2 \leq t$ and then by iterating to all t. We got:

<u>Prop</u> The semigroups $S_\Lambda(t)$ converge to $S(t)$

$$\| S_\Lambda(t)f - S(t)f \| \xrightarrow[\Lambda \nearrow \mathbb{Z}^d]{} 0$$

$S(t)$ is a strongly continuous semigroup with generator $\tilde{L}$ such that $\tilde{L}\restriction_{\mathcal{C}} = L$.

<u>Remark</u> We have bypassed the Hille-Yoshida. What is still missing is uniqueness?

<u>Prop</u> Suppose $S(t)$ and $S'(t)$ two semigroups with generators $\tilde{L}, \tilde{L}'$, $\tilde{L}\restriction_{\mathcal{C}} = \tilde{L}'_{\mathcal{C}} = L$. Then they agree: $S(t) = S'(t)$.

<u>Pf</u> Enough to show $S(t)f = S'(t)f$ $\forall f \in \mathcal{C}$. and $t \le t_0$. Let $f \in \mathcal{C}_\Lambda$. Then since $L^n f \in \mathcal{C}$ $\forall n$,

$$S(t)f = \sum_{n=0}^{N} \frac{t^n}{n!} L^n f + \frac{1}{N!} \int_0^t (t-s)^N S(s) L^{N+1} f \, ds$$

$$S'(t)f = \sum_{n=0}^{N} \frac{t^n}{n!} L^n f + \frac{1}{N!} \int_0^t (t-s)^N S'(s) L^{N+1} f \, ds$$

$\Rightarrow$

$$\| S(t)f - S'(t)f \| \le \frac{2}{N!} t^{N+1} \| L^{N+1} f \|$$

$$< 2(N+1)\, e^{|\Lambda|} \|f\| (c\,t)^{N+1} \quad \forall N$$

So for $t \le t_0$. $S(t)f = S'(t)f$ $\forall f \in \mathcal{C}$ $\square$

Next we show a crude bound of $S(A)$ preserving locality. <u>We assume $d=1$.</u>

<u>Prop</u> Let $\mathcal{E} \subset C(X)$ be the set

$$\mathcal{E} = \{ f \mid \exists A<\infty,\ b>0 : \sup_{\eta\restriction_{\Lambda_\ell} = \eta'\restriction_{\Lambda_\ell}} |f(\eta)-f(\eta')| \le A e^{-b\ell} \ \forall \ell \text{ and } \|f\| \le 1 \}$$

where $\Lambda_\ell = [-\ell, \ell] \subset \mathbb{Z}$. Then, $S(t): \mathcal{E} \to \mathcal{E}$.

<u>Pf</u> Let $\sup |f(\eta)-f(\eta')| \le A e^{-b\ell}$, $\forall \ell$.

Let $f_\ell(\eta) = f(\eta_\ell)$ where

$$\eta_\ell(x) = \begin{cases} \eta(x) & |x| \le \ell \\ 0 & |x| > \ell \end{cases}$$

Write $f = f_\ell + f - f_\ell$. Then

$$\| S(t)(f-f_\ell) \| \le A e^{-b\ell}$$

As above we have

$$\| S(t) f_\ell - S_{\Lambda_{2\ell}}(t) f_\ell \| \le 2\|f_\ell\| e^{|\Lambda_\ell|} \cdot \sum_{n \ge c\ell} (ct)^n \le 2 e^{2\ell} (c t)^{c\ell}$$

Since $(S_{2\ell}(t) f_\ell)(\eta)$ depends on η in $[-2\ell - R, 2\ell + R]$ we get, for $\ell > R$

$$\sup_{\eta\restriction_{\Lambda_{3\ell}} = \eta'_{\Lambda_{3\ell}}} |S(t)f(\eta) - S(t)f(\eta')| \le A e^{-b\ell} + (ct)^{c\ell}$$

For $l < R$ get $\leq 2\|f\| \leq 2$.

Hence $S(t)f$ satisfies the claim

with $b \to b/3$, $A \to A + C$ $\square$

Corollary We get by iteration (put

$t = n t_0$,

$$\sup_{\eta\restriction_{\Lambda_l} = \eta'\restriction_{\Lambda_l}} |(S(t)f)(\eta) - (S(t)f)(\eta')| \leq A(t)\, e^{-b(t)\, l}$$

with $b(t) \geq e^{-ct}$, $A(t) \leq Ct$.

Apply to $S(t,x) = E^{\mu}(\eta_t(x)\eta_0(0))$. Lets consider

the ∞ temperature case, i.e. μ = Bernoulli measure

In general we have

$$E^{\mu}(f(\eta_t)\, g(\eta_0)) = \int \mu(d\eta)\, E^{\eta}(f(\eta_t)\, g(\eta))$$

$$= \int \mu(d\eta)\, (S(t)f)(\eta)\, g(\eta)$$

Write $S(t,x) = E^{\mu}((\eta_t(0) - \rho)(\eta_0(-x) - \rho))$

and take $f(\eta) = \eta(0)$, $g(\eta) = \eta(-x)$ to get

$$S(t,x) = E^{\mu}(((S(t)f)(\eta) - \rho)(\eta(-x) - \rho))$$

Write $S(t)f = (S(t)f)_l + S(t)f - (S(t)f)_l$

for $l = |x| - 1$. Then $S(t)f_l$ is independent

on $\eta(-x)$ so since $E^{\mu}(\eta(-x) - \rho) = 0$ get

$$|S(t,x)| \leq 2\, \| S(t)f - (S(t)f)_l \| \leq A(t)\, e^{-b(t)|x|} \quad \square$$

More work gives also $d > 1$.

This bound is LOUSY as we'll see below.

<u>Remark</u> For $\beta > 0$ need to know that correlations in μ-Gibbs state decay exponentially: Let $f \in \mathcal{C}_{\Lambda_1}$, $g \in \mathcal{C}_{\Lambda_2}$

$$\left| E^\mu f g - E^\mu f \, E^\mu g \right| \leq C e^{-a \operatorname{dist}(\Lambda_1, \Lambda_2)} \qquad a > 0.$$

Let

$$\hat{S}(t, k) = \sum_{x \in \mathbb{Z}^d} S(t, x) e^{-ikx} \qquad \left(kx = \sum_{\alpha=1}^{d} k_\alpha x_\alpha\right)$$

be the Fourier transform, converges due to exponential decay. Then

$$\hat{S}(t, k) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \frac{1}{|\Lambda|} \sum_{x, y \in \Lambda} S(t, x-y) e^{-ik(x-y)}$$

$$= \lim \frac{1}{\Lambda} \sum_{x, y \in \Lambda} E^\mu (\eta_t(x) - \rho)(\eta_0(y) - \rho) e^{-ik(x-y)}$$

$$= \lim_{\Lambda} E^\mu \left(\overline{f}_{\Lambda,k} S(t) f_{\Lambda,k} \right)$$

$$\left(f_{\Lambda,k}(\eta) = \frac{1}{\sqrt{|\Lambda|}} \sum_{y \in \Lambda} e^{-iky} (\eta(y) - \rho) \right)$$

$$= \lim_{\Lambda \nearrow \mathbb{Z}^d} \left(f_{\Lambda,k}, S(t) f_{\Lambda,k} \right)$$

scalar product $L^2(\mu)$.