Invariant measures

Let P^X be Feller Markov, $S(t)$ semigroup.

$S(t): C_b(X) \to C_b(X)$.

S induces an action on $\mathcal{M}_1(X) = \{$prob. measures on $X\}$: Let $\mu \in \mathcal{M}_1(X)$, set

$$S^*(t)\mu \in \mathcal{M}_1 \quad \text{as}$$

$$\int f \, d(S^*(t)\mu) = \int S(t) f \, d\mu \qquad \forall f \in C_b(X)$$

or,

$$(S^*(t)\mu)(A) = \int P^x(X_t \in A)\, \mu(dx)$$

$\forall$ Borel set A.

Def μ is <u>invariant</u> if $S^*(t)\mu = \mu \quad \forall t \geq 0$.

Give X_0 prob. distribution μ. Then

X_t has — " —

(X_t) is <u>stationary</u> $(X_t)_{t=0}^{\infty} \simeq (X_{s+t})_{t=0}^{\infty}$

<u>Q</u> · What are invariant μ? Existence, uniqueness
· given $\nu \in \mathcal{M}_1$, does $S^*(t)\nu \underset{t\to\infty}{\Rightarrow} \mu$?

<u>Thm</u> Let V be a core for L. Then

$\mu \in \mathcal{M}_1$ is invariant $\iff$

$$\int Lf \, d\mu = 0 \qquad \forall f \in V$$

<u>Pf</u> "$\Rightarrow$" Let $f \in \mathcal{D}(L)$. So

$$\lim_{t\to 0} \sup_{x \in X} \left| \frac{(S(t)f)(x) - f(x)}{t} - Lf(x) \right| = 0 \quad \text{so}$$

$$\int Lf \, d\mu = \lim_{t \to 0} \underbrace{\int \frac{S(t)f - f}{t} \, d\mu}$$

$$= \int \frac{f - f}{t} \, d\mu = 0$$

"$\Leftarrow$" Let $f \in \mathcal{D}(L)$. $\exists f_n \in V$, $\|f_n - f\| \to 0$

and $\|Lf_n - Lf\| \to 0 \Rightarrow \int Lf \, d\mu = \lim \int Lf_n \, d\mu$

$= 0$. So $\int Lf \, d\mu = 0$. Thus, since $S(t)f \in \mathcal{D}(L)$

and $S(t)f - f = \int_0^t LS(s)f \, ds \Rightarrow$

$$\int (S(t)f - f) \, d\mu \underset{\uparrow}{=} \int_0^t ds \int L(S(s)f) \, d\mu = 0$$

Fubini

$\Rightarrow \int S(t)f \, d\mu = \int f \, d\mu \quad \forall f \in \mathcal{D}(L)$

$\mathcal{D}(L)$ dense $\Rightarrow$ $\uparrow$ holds $\forall f \in C_b(X)$ $\square$

For exclusion process we have the nice

<u>Prop.</u> $\mathcal{C} = \{$Cylinder functions$\}$ = core for L.

Idea of Pf. 1. We will 1st find an invariant subspace $D \subset C(X)$:

$$S(t) D \subset D$$

There are many choices

For $f \in C(X)$ let $(\partial_x f)(\eta) = f(\eta^1) - f(\eta^2)$

where $\eta^1(y) = \eta^2(y) = \eta(y) \quad y \neq x$

$\eta^1(x) = 1, \ \eta^2(x) = 0$

Take

$$D = \{ f \in C(X) \mid \sum_{x \in \mathbb{Z}^d} \| \partial_x f \|_\infty < \infty \}$$

Note: $\mathcal{C} \subset D$ since $\partial_x f \neq 0$ only for finite # of x.

Let $t \leq t_0$, $C_t(x)$ c.c. of $\mathcal{G}_t$ containing x.

$$|(\partial_x S(t) f)(\eta)| = |E(f(\eta_t^{\eta^1}) - f(\eta_t^{\eta^2}))|$$

$$\leq E|f(\eta_t^{\eta^1}) - f(\eta_t^{\eta^2})|$$

$$\leq E \sum_{y \in C_t(x)} \| \partial_y f \|_\infty \qquad (*)$$

We used $\eta_t^{\eta^1}(z) = \eta_t^{\eta^2}(z) \quad z \notin C_t(x)$

and
$$|f(\eta) - f(\zeta)| \leq \sum_{x : \eta(x) \neq \zeta(x)} \| \partial_x f \|_\infty$$

Now

$$(*) = \sum_y \| \partial_y f \|_\infty \, P(y \in C_t(x))$$

Hence, since $y \in C_t(x) \Leftrightarrow x \in C_t(y)$ get

$$\sum_x \| \partial_x S(t) f \|_\infty \leq \sum_y \| \partial_y f \|_\infty \sum_x P(x \in C_t(y))$$

But $\sum_x P(x \in C_t(y)) = E|C_t(y)|$

$= E|C_t(0)|$ by transl. invariance

so claim follows if

$$E|C_t(0)| < \infty$$

Now if $|C_t(0)| \geq N$ there is a path from 0 to y with $|y| > cN^{1/d}$. So

$$P(|C_t(0)| \geq N) \leq C\lambda^{cN^{1/d}} \qquad \lambda < 1$$

so $\sum_N$ — " — $< \infty$ $\square$.

2. Next we observe: if $f \in D$ then

$$|Lf| = |\sum_{x,y} p(x,y)\eta(x)(1-\eta(y))(f(\eta^{xy}) - f(\eta))|$$

$$\leq \sum_{x,y} p(x,y)(\|\partial_x f\|_\infty + \|\partial_y f\|_\infty) < \infty$$

To prove $D \subset \mathcal{D}(L)$ need to show still

$$\frac{S(t)f - f}{t} \xrightarrow[t \to 0]{} Lf \qquad f \in D.$$

3. To show this, given $f \in D$ show $\exists f_n \in \mathcal{C}$, $f_n \Rightarrow f$ $Lf_n \Rightarrow Lf$. Then, since L is closed $f \in \mathcal{D}(L)$. This approximation by cylinder is not hard.

4. So we got

$$\mathcal{C} \subset D \subset \mathcal{D}(L) \quad , \quad \mathcal{C}, D \text{ dense}$$

$$S(t) : D \to D$$

The Prop. on p. 59 $\Rightarrow$ D is a core

But 3. shows $\overline{\text{Graph}(L\restriction_{\mathcal{C}})} \supset \text{Graph}\, L\restriction_D$

so $\mathcal{C}$ is also a core $\square$.

Bernoulli measures

Def Bernoulli measure, density ρ $(0 \le \rho \le 1)$ ν_ρ is the measure where $\eta(x)$ are i.i.d mean ρ. I.e.

$$\nu_\rho = \prod_{x \in \mathbb{Z}^d} \nu^0_\rho(d\eta(x))$$

ν^0_ρ measure on $\{0,1\}$ $\quad \nu^0_\rho(0) = 1-\rho, \; \nu^0_\rho(1) = \rho$

So

$$\nu_\rho(\eta : \eta(x) = 1 \;\; x \in A, \; \eta(y) = 0, \; y \in B)$$
$$= \rho^A (1-\rho)^B$$

Prop ν_ρ is invariant for translation invariant exclusion process

Pf Show $\int Lf \, d\nu_\rho = 0$ $\quad f$ cylinder

f is a linear combination of $\prod_{x \in A} \eta(x) \equiv \eta_A$

[proof: Let $g : \{0,1\} \to \mathbb{R}$. Then

$g(\eta) = g(0)(1-\eta) + g(1)\eta$, iterate]

Compute
$$L\eta_A = \sum_{x,y} \underbrace{\eta(x)(1-\eta(y))\, p(x,y) \left[(\eta^{xy})_A - \eta_A \right]}_{(*)}$$

If $x, y \in A$ then $(\eta^{xy})_A = \eta_A$ get 0.
$\quad x, y \in A^c$

$x \in A, \; y \in A^c$: $\quad (*) = \eta(x)(1-\eta(y)) \left[\eta(y) - \eta(x) \right] \eta_{A \setminus x} \, p(x,y)$
$$= -(1-\eta(y))\, \eta_A \, p(x,y)$$

$x \notin A \quad y \in A \quad (*) = \eta(x)(1-\eta(y))[\eta(x)-\eta(y)]\,\eta_{A+y}\,p(x,y)$

$$= \eta(x)(1-\eta(y))\,\eta_{A+y}\,p(x,y)$$

$$= (\eta(x)\,\eta_{A+y} - \eta(x)\,\eta_A)\,p(x,y)$$

$$\int L\eta_A\,d\nu = \sum_{x\in A,\, y\in A^c} (g-1)g^A\, p(x,y)$$

$$+ \sum_{x\in A^c,\, y\in A} (g^A - g^{A+1})\, p(x,y)$$

$$= (g-1)g^A \sum_{x\in A,\, y\in A^c} (p(x,y) - p(y,x))$$

Since $p(x,y) = \tilde{p}(x-y)$ the sum is

$$\sum_{x\in A}\sum_{y\in A^c} (\tilde{p}(x-y) - \tilde{p}(y-x))$$

$$= \underbrace{\sum_{x\in A}\sum_{y\in \mathbb{Z}^d} (\tilde{p}(x-y)-\tilde{p}(y-x))}_{\substack{=0 \\ \text{since } \sum_z \tilde{p}(z)=0}} - \underbrace{\sum_{\substack{x\in A\\ y\in A}} (\tilde{p}(x-y)-\tilde{p}(y-x))}_{\substack{=0 \\ \text{symmetry } x\leftrightarrow y}}$$

$$= 0 \quad \square$$

Other processes

Let us generalize a bit. Consider the dynamics on $\{0,1\}^{\mathbb{Z}^d}$ with generator

$$(Lf)(\eta) = \frac{1}{2}\sum_{x,y} c(x,y,\eta)\,(f(\eta^{xy}) - f(\eta))$$

Since $\eta^{xy} = \eta^{yx}$ we may assume

(a) $c(x,y,\eta) = c(y,x,\eta)$

Assume also

(b) $c(x,y,\eta) = 0$ if $|x-y| > R$.

Translation invariance assumption:

(c) $c(x,y,\eta) = c(x+z, y+z, \tau_z \eta)$

$$(\tau_z \eta)(x) \equiv \eta(x-z)$$

How to construct $S(t)$ and process η_t now?

Recall what we did for exclusion:

- We wrote $\eta_t(\omega)$, $\omega \in \Omega$ of Poisson, explicite construction
- We checked explicitely that

$$P^{\eta}(A) = P_{POISSON}(\{\eta^{\eta}(\omega) \in A\})$$

is Feller process.

- Thus it has generator $\tilde{L}$ and on $\mathcal{C}$ generator is our L: $\tilde{L}\restriction_{\mathcal{C}} = L$.

Could there be another Markov process $\tilde{P}^\eta$ on $D_X[0,\infty)$, $\tilde{P}^\eta \neq P^\eta$ but with the same generator?

<u>Prop</u> Suppose $\tilde{P}^\eta$ is a Markov process on $D_X[0,\infty)$ with:

- semigroup $\tilde{S}(t) = \tilde{E}^\eta(f(\eta_t))$ is strongly continuous on $C(X)$
- Generator $\tilde{L}$ of $\tilde{S}$ has $\mathcal{C} \subset \mathcal{D}(\tilde{L})$ and $\tilde{L}\restriction_{\mathcal{C}} = L$ ($\mathcal{C}$ = cylinder)

Then $\tilde{P}^\eta = P^\eta \quad \forall \eta \in X$.

<u>Pf</u> Both $\tilde{L}$ and L are closed.

Then $\mathcal{D}(L) \subset \mathcal{D}(\tilde{L})$:

Let $f \in \mathcal{D}(L)$. Then ($\mathcal{C}$ is core for L) $\exists f_n \in \mathcal{C}$

$$f_n \to f, \quad Lf_n \to Lf$$

But $Lf_n = \tilde{L}f_n$ so $\tilde{L}f_n$ converges. Hence since $\tilde{L}$ is closed $f \in \mathcal{D}(\tilde{L})$ and $\tilde{L}f = Lf$.

So $\tilde{L}$ is an extension of L

We need:

<u>Lemma</u> Suppose $L, \tilde{L}$ are generators of strongly

continuous contraction semigroups and $\tilde{L}$ is an extension of L. Then $\tilde{L} = L$ and $\tilde{S}(t) = S(t)$ $\forall t$.

Proof See Seppäläinen Lectures □

Thus transition probabilities

$$p_t(\eta, A) = P^\eta(\eta_t \in A) = (S(t)\,\mathbb{1}_A)(\eta)$$

agree: $p_t = \tilde{p}_t$. Hence $\tilde{P}^\eta$ and P^η agree on cylinder sets of $D_X[0,\infty)$ and then everywhere. □

How to proceed when only $L\restriction_{\mathcal{C}}$ is given as above?

Positivity

How is the contraction of $S(t)$ reflected in L?

Def Let $L: \mathcal{D}(L) \to B$, B Banach space

Resolvent set $\varrho(A) = \{\lambda \in \mathbb{C} \mid \exists (\lambda - A)^{-1}\}$

i.e. $\lambda \in \varrho(A)$ if $\lambda - A : \mathcal{D}(A) \to B$ is 1-1 and onto and $(\lambda - A)^{-1} : B \to \mathcal{D}(A)$ is bounded. For $\lambda \in \varrho(A)$, $(\lambda - A)^{-1}$ is the resolvent of A.

<u>Prop</u> Let L be the generator of strongly continuous contraction semigroup $S(t)$. Then $(0,\infty) \in \rho(L)$ i.e. $\exists\, (\lambda - L)^{-1}$ if $\lambda > 0$ and

$$(\lambda - L)^{-1} f = \int_0^\infty e^{-\lambda t} S(t) f \, dt$$

<u>Pf</u> Since $\|S(t) f\| \le \|f\|$ operator

$$B_\lambda = \int_0^\infty e^{-\lambda t} S(t) \, dt$$

is bounded:

$$\| B_\lambda f \| \le \int_0^\infty e^{-\lambda t} dt \, \|f\| \le \lambda^{-1} \|f\|$$

[actually B_λ bounded if $\mathrm{Re}\,\lambda > 0$ and $\lambda \to B_\lambda$ is analytic there].

$$\frac{S(\tau) - 1}{\tau} B_\lambda f = \frac{1}{\tau} \int_0^\infty e^{-\lambda t} (S(t+\tau) - S(t)) f dt$$

$$= \frac{e^{\lambda \tau}}{\tau} \int_\tau^\infty e^{-\lambda t} S(t) f dt - \frac{1}{\tau} \int_0^\infty e^{-\lambda t} S(t) f dt$$

$$= \frac{e^{\lambda \tau} - 1}{\tau} \int_0^\infty e^{-\lambda t} S(t) f dt - \frac{1}{\tau} \int_0^\tau e^{-\lambda t} S(t) f dt$$

$$\xrightarrow[\tau \to 0]{} \lambda B_\lambda f - f$$

so $B_\lambda f \in \mathcal{D}(L)$ $\forall f \in B$ and $L B_\lambda f = \lambda B_\lambda f - f$

i.e. $(\lambda - L) B_\lambda f = f$.

Next, let $f \in \mathcal{D}(L)$. Then

$$B_\lambda L f = \int_0^\infty e^{-\lambda t} S(t) L f \, dt = \int_0^\infty L \big(\underbrace{e^{-\lambda t} S(t) f}_{\in \mathcal{D}(L)} dt \big)$$

$$\underset{\text{check!}}{=} L \int_0^\infty e^{-\lambda t} S(t) f \, dt = L B_\lambda f$$

So got $\forall f \in B \quad (\lambda - L) B_\lambda f = f \qquad (1)$

$\forall f \in \mathcal{D}(L) \quad B_\lambda (\lambda - L) f = f \qquad (2)$

in (1) : $\lambda - L$ is onto & $(\lambda - L)^{-1} = B_\lambda$

(2) : $\lambda - L$ is 1-1

and $(\lambda - L)^{-1}$ is bounded sing B_λ is □

<u>Corollary</u> $\forall f \in \mathcal{D}(L)$, $\lambda > 0$

$$\| (\lambda - L) f \| \geq \lambda \| f \|$$

<u>Pf</u> $\| (\lambda - L)^{-1} g \| \leq \lambda^{-1} \| g \|$. Take $g = (\lambda - L) f$ □

<u>Def</u> Let $B = C(X)$. $L : \mathcal{D}(L) \to C(X)$

is a Markov <u>pre-generator</u> if

a) $1 \in \mathcal{D}(L)$, $L1 = 0$

b) $\mathcal{D}(L)$ is dense

c) If $f \in \mathcal{D}(L)$, $\lambda > 0$ and $(\lambda - L) f = g$

Then $\lambda \cdot \min_{\eta \in X} f(\eta) \geq \min_{\eta \in X} g(\eta)$

Note apply c) to $-f$ to get

$$\lambda \max f \leq \max g$$

$\Rightarrow \quad \lambda \| f \| \leq \| (\lambda - L) f \|$.

Now (Exercise)

<u>Prop</u> a) Let L be a Markov pre generator. Then L has a closure $\bar{L}$ which also is.

b) Let L be closed Markov pre-generator. Then range of $\lambda - L$ is a closed subspace of $C(X)$

<u>Def</u> A <u>Markov generator</u> is a closed Markov pre generator L s.t. Range of $(\lambda - L) = C(X)$ for λ large enough

<u>Remark</u> This $\Rightarrow$ $\text{Range}(\lambda - L) = C(X) \quad \forall \lambda > 0$

Indeed Set $\mu < \lambda$, $g \in C(X)$. Try to solve

$$(\mu - L) f = g$$

Define $T : C(X) \to \mathcal{D}(L)$

$$Th = (\lambda - L)^{-1} g + (\lambda - \mu)(\lambda - L)^{-1} h$$

Note $\| (\lambda - \mu)(\lambda - L)^{-1} h \| \leq \frac{\lambda - \mu}{\lambda} \|h\|$

and $\frac{\lambda - \mu}{\lambda} < 1$ so $\| (\lambda - \mu)(\lambda - L)^{-1} \| < 1$

so $1 - (\lambda - \mu)(\lambda - L)^{-1}$ is invertible

so $\exists f : Tf = f$ namely

$$f = (1 - (\lambda - \mu)(\lambda - L)^{-1})^{-1} (\lambda - L)^{-1} g$$

This is our f $\square$

The main result is

<u>Hille Yoshida Theorem</u> There is 1-1 correspondence with Markov generators on $C(X)$ and Markov semigroups:

<u>Given $S(t)$</u>, L, $\mathcal{D}(L)$ as before

<u>Given L</u>, $S(t)f = \lim_{n\to\infty} \left(1 - \frac{t}{n} L\right)^{-n} f$ $\quad t \geq 0$, $f \in C(X)$

To apply this to our case, we need to show:

a) L is a pre-generator
b) $\mathrm{Ran}(1-L)$ is dense + large.

a) is easy:

1. $L1 = 0$ clear

2. $\mathcal{D}(L)$ dense. The set

$$D = \{ f : \sum_x \|\partial_x f\|_\infty \equiv |||f||| < \infty \}$$

satisfies $L: D \to C(X)$

<u>Proof</u> $|f(\eta^{xy}) - f(\eta)| \leq \|\partial_x f\|_\infty + \|\partial_y f\|_\infty$

so $\left| \sum_{x,y} c(x,y,\eta)\left(f(\eta^{xy}) - f(\eta)\right) \right|$

$$\leq \sum_{x,y} c(x,y,\eta) \left(\|\partial_x f\|_\infty + \|\partial_y f\|_\infty \right)$$

$$\leq C |||f||| \quad \square$$

3. How about positivity?

<u>Lemma</u> Suppose L satisfies:

(*) $\forall f \in \mathcal{D}(L)$: If $f(\eta_0) = \min_\eta f(\eta)$ then $(Lf)(\eta_0) \geq 0$

Then $\forall f \in \mathcal{D}(L)$:

$$\lambda \min f \geq \min (\lambda - L) f$$

<u>Pf</u> $\lambda \min f = \lambda f(\eta_0) \geq \lambda f(\eta_0) - Lf(\eta_0) \geq \min(\lambda - L) f$ $\square$

Thus suffices to check (*). This is easy:

Let $f(\eta_0) = \min f(\eta)$. Then $f(\eta_0^{xy}) \geq f(\eta_0)$

$\Rightarrow Lf(\eta_0) \geq 0$.

b) Is much harder. See Liggett's book.

<u>Upshot</u>: Our L has $\mathcal{C}$ as a core and gives rise to a continuous Markov semigroup $S(t)$ on $C(X)$. Also,

$$S(t): D \to D$$

<u>Reversibility</u>

<u>Equilibrium states of lattice gases</u>:

Let us work in finite volume $\eta \in \{0,1\}^\Lambda = X_\Lambda$

$\Lambda \subset \mathbb{Z}^d$ $|\Lambda| < \infty$, say $\Lambda = \mathbb{T}_N^d$.

Equilibrium state: Let $\mathcal{H}_\Lambda : X_\Lambda \to \mathbb{R}$

be "energy" of η. <u>Gibbs measure</u> $\mu_\Lambda \in \mathcal{M}_1(X_\Lambda)$

$$\mu_\Lambda^\beta(\eta) = e^{-\beta H_\Lambda(\eta)}/Z \qquad Z = \sum_\eta e^{-\beta H_\Lambda(\eta)}$$

When is μ_Λ^β invariant : $S^*(t)\mu_\Lambda^\beta = \mu_\Lambda^\beta$?

We say $\mu \in \mathcal{M}_1(X_\Lambda)$ is <u>reversible</u> if it is invariant i.e. $S^*(t)\mu = \mu$ and invariant under time reversal : Let P^μ be the measure on $D_X(0,\infty)$ $P^\mu(\cdot) = \int \mu(d\eta) P^\eta(\cdot)$

Then want

$$E^\mu\left(\prod_{i=1}^n f_i(\eta_{t_i})\right) = E^\mu\left(\prod f_i(\eta_{T-t_i})\right)$$

for all $f_i \in C(X)$, $t_1 < t_2 \dots < t_n < T$, $\forall T$.

$$\Leftrightarrow \quad P^\mu(\eta_0 = \eta, \eta_t = \eta') = P^\mu(\eta_0 = \eta', \eta_t = \eta)$$

$$\Leftrightarrow \quad \mu(\eta) p_t(\eta, \eta') = \mu(\eta') p_t(\eta', \eta) \qquad (*)$$

Measure μ satisfying $(*)$ is <u>reversible</u>.

Now

$$\frac{d}{dt} \sum_{\eta'} p_t(\eta, \eta') f(\eta') = (L\, S(t) f)(\eta)$$

$$= \sum c(x, y, \eta)(p_t(\eta^{xy}, \eta') - p_t(\eta, \eta')) f(\eta')$$

i.e.

$$\frac{d}{dt}\Big|_{t=0} p_t(\eta, \eta') = \sum c(x, y, \eta)(p_t(\eta^{xy}, \eta') - p_t(\eta, \eta'))\Big|_{t=0}$$

$$= \sum_{xy} c(x, y, \eta)(\delta_{\eta^{xy}\eta'} - \delta_{\eta\eta'})$$

so $(*)$ $\Leftrightarrow$

$$\mu(\eta) c(x, y, \eta) = \mu(\eta^{xy}) c(x, y, \eta^{xy})$$

i.e. if $\mu(\eta) = \frac{1}{Z} e^{-\beta H(\eta)}$ we get reversibility $\Leftrightarrow$ detailed balance:

$$(**) \quad c(x,y,\eta) = c(x,y,\eta^{xy}) \exp[-\beta H(\eta^{xy}) + \beta H(\eta)]$$

Thus $(**) \Rightarrow \frac{1}{Z} e^{-\beta H}$ is invariant measure and system is reversible in time.

Let $\mathcal{H} = L^2(X_\Lambda, \mu)$ i.e. scalar product

$$(f,g) = \sum_{\eta \in X_\Lambda} f(\eta)\, \overline{g(\eta)}\, \mu(\eta)$$

Then μ reversible $\Leftrightarrow$ L self adjoint.

$$(f, Lg) = (Lf, g) \qquad (1)$$

Indeed $(1) \Leftrightarrow L(\eta,\eta')\mu(\eta) = L(\eta',\eta)\mu(\eta')$ which is reversibility (L is real).

Infinite volume

Finite range interaction: Let $J_A \in \mathbb{R}$, $A \subset \mathbb{Z}^d$ s.t.

a) $J_A = 0$ diameter$(A) > R$, $R < \infty$.

b) $J_{a+A} = J_A$ $a \in \mathbb{Z}^d$

This data determines Hamiltonian (energy) in region $\Lambda \subset \mathbb{Z}^d$:

$$H_\Lambda(\eta) = \sum_{A \subset \Lambda} J_A\, \eta(A) \qquad \eta(A) = \prod_{x \in A} \eta(x)$$

Reversibility — as volume

Def μ reversible if $\forall f, g \in C(X)$

$$\int f\, S(t) g \, d\mu = \int (S(t) f)\, g \, d\mu$$

So $f = 1 \Rightarrow \mu$ invariant

Prop μ reversible $\Leftrightarrow$ $\int f L g \, d\mu = \int Lf \, g \, d\mu$

$\forall f, g \in \mathcal{D}(L)$

$\Leftrightarrow$ same $\forall f, g \in$ Core

Pf "$\Rightarrow$" $\int f \, \frac{S(t)g - g}{t} \, d\mu = \int \frac{S(t)f - f}{t} \, g \, d\mu$

take $t \to 0$

"$\Leftarrow$" let $\lambda \geq 0$, $\tilde{f}, \tilde{g} \in \mathcal{D}(L)$ then

$$\int \tilde{f} (1 - \lambda L) \tilde{g} \, d\mu = \int \tilde{g} (1 - \lambda L) \tilde{f} \, d\mu$$

Take $\tilde{f} = (1 - \lambda L)^{-1} f$, $\tilde{g} = (1 - \lambda L)^{-1} g$ $\Rightarrow$

$$\int (1 - \lambda L)^{-1} f \, g \, d\mu = \int (1 - \lambda L)^{-1} g \, f \, d\mu$$

Put $\lambda = t/n$ and iterating get

$$\int \left(1 - \tfrac{t}{n} L\right)^{-n} f \, g \, d\mu = \int \left(1 - \tfrac{t}{n} L\right)^{-n} g \, f \, d\mu$$

$\forall t \geq 0$, n, $f, g \in C(X)$. Hille-Yoshida $\Rightarrow$ Ŋ

Let ν be Bernoulli $g = 1/2$ measure on $X = \{0,1\}^{\mathbb{Z}^d}$

and

$$\mu_\Lambda^\beta = \frac{1}{Z_\Lambda} e^{-\beta H_\Lambda} \nu$$

Then one can prove:

$$\mu_\Lambda^\beta \longrightarrow \mu^\beta \quad \text{weakly as } \Lambda \uparrow \mathbb{Z}^d$$

μ^β is a <u>Gibbs measure</u> corresponding to (J_A).

[For β small this is unique i.e. indep. on the way we define H_Λ (boundary conditions)

For β large there can be <u>phase transitions</u>]

<u>Example</u> $J_{\{x\}} = \lambda$, $J_{\{x,y\}} = J$ $|x-y| = 1$

other $J_A = 0$. So

$$H_\Lambda = \lambda \sum_{x \in \Lambda} \eta(x) + J \sum_{|x-y|=1} \eta(x)\eta(y)$$

is <u>Ising model</u>. [usually one defines <u>spin</u> variable $\sigma = 2\eta - 1 \in \{-1, 1\}$ so

$$H = \left(\frac{\lambda}{2} + \frac{d}{2} J\right) \sum \sigma_x + \frac{J}{4} \sum \sigma_x \sigma_y$$

For $J < 0$ (ferromagnet) and $\beta |J|$ large there are two phases if $\lambda + dJ = 0$ (i.e. for critical fugacity $e^{-\beta\lambda}$):

$$E\,\sigma(x) = \pm m \quad , m > 0 \quad \text{i.e.} \quad E\,\eta(x) = \frac{1}{2} \pm \frac{m}{2}$$

i.e. low and high density (gas and liquid)

[For $\beta \to 0$ (infinite temperature) μ is Bernoulli $\frac{1}{2}$

We get other Bernoulli by $h = \frac{\alpha}{\beta}$ i.e.

$$e^{-\beta H} \underset{\beta \to 0}{\longrightarrow} e^{-\alpha \sum \eta(x)} \; ; \quad E\,\eta(x) = \tfrac{1}{2} e^{-\alpha} \; . \;]$$

<u>Detailed balance</u>: Note that

$$H_\Lambda(\eta^{xy}) - H_\Lambda(\eta) = \sum_{\substack{A:\, x\in A \\ y \notin A}\ \text{or}\ \substack{y \in A \\ x \notin A}} J_A \left(\eta_A^{xy} - \eta_A\right)$$

Has a limit as $\Lambda \uparrow \mathbb{Z}^d \equiv \Delta_{xy} H(\eta)$

Thus require

$$(*) \qquad c(x,y,\eta) = c(x,y,\eta^{xy})\, e^{-\beta \Delta_{xy} H(\eta)}$$

If $(*)$ holds then μ^β is an invariant measure for the Markov process.

<u>Remark</u> Thus e.g. Ising process has for β large <u>two</u> (ergodic) invariant measures corresponding to small and large density.

<u>Example</u> $(*)$ holds, if take

$$c(x,y,\eta) = F(\Delta_{xy} H(\eta))$$

s.t. $F(E) = F(-E)\, e^{\beta H}$

Examples of this are

Kawasaki dynamics $\quad F(E) = (1 + e^{\beta E})^{-1}$

Metropolis —"— $\quad F(E) = \begin{cases} 1 & E \leq 0 \\ e^{-\beta H} & E > 0 \end{cases}$

In Metropolis, if $\Delta_{xy} H(\eta) \leq 0$ then $c = 1$ i.e. jump, if > 0 then with probability $e^{-\beta \Delta_{xy} H}$ jump, $1 - e^{-\beta \Delta_{xy} H}$ do not jump.

<u>Note</u> The chemical potential $\lambda = J_{\{x\}}$ will not contribute to $\Delta_{xy} H$ so if detailed balance holds, there is (at least) a 1-parameter family of invariant measures

<u>Invariance & generator</u>

μ invariant $\Leftrightarrow \int Lf \, d\mu = 0 \quad \forall f \in \mathcal{C}$

μ reversible $\Leftrightarrow \int f Lg \, d\mu = \int g Lf \, d\mu \quad \forall f, g \in \mathcal{C}$

i.e. $(f, Lg) = (Lf, g)$

$\Leftrightarrow \int c(x,y,\eta) [f(\eta^{xy}) - f(\eta)] \, d\mu = 0$

$\forall f \in \mathcal{C}$

Given L (i.e. $c(x,y,\eta)$) what is the set of invariant measures?

If the rates satisfy detailed balance with some Hamiltonian as above, then translation-invariant invariant measures are Gibbs measures

Ex Exclusion process

$$c(x,y,\eta) = p(x,y)\eta(x)(1-\eta(y)) + p(y,x)\eta(y)(1-\eta(x))$$

Symmetric exclusion
$p(x,y) = p(y,x)$ is reversible w.r.t. Bernoulli (i.e. $\beta = 0$)

$$c(x,y,\eta) = c(x,y,\eta^{xy})$$

If $p(x,y) = p(x-y)$ every invariant measure is a convex combination of Bernoullis.

Ex Asymmetric exclusion, transl. invariant

Bernoulli are invariant but not reversible.

We can think about these as driven systems

Ex 1 dimensional case, nearest neighbour (i.e. simple asymmetric exclusion)

Add a constant "Force" to Hamiltonian

$$H(\eta) \to H(\eta) - \sum_x F x \, \eta(x) \qquad (*)$$

i.e. particle at $x \in \mathbb{Z}$ has potential energy $-Fx$ $\quad$ (Force $= -\frac{\partial V}{dx} = F$)

Demand detailed balance w.r.t. (*)

$$c(x,y,\eta) = c(x,y,\eta^{xy})\, e^{-\beta \Delta_{xy} H - F(x-y)(\eta(x)-\eta(y))}$$

For ASEP $H=0$ and $g=1$

$$C(x,x+1,\eta) = p\,\eta(x)(1-\eta(x+1)) + (1-p)\,\eta(x+1)(1-\eta(x))$$

i.e. jump to right prob p, left $1-p$.

Get $C = p\,\eta(x) + (1-p)\,\eta(x+1)$ so

$$p\,\eta(x) + (1-p)\eta(x+1) = (p\,\eta(x+1) + (1-p)\,\eta(x))\,e^{\beta F(\eta(x)-\eta(x+1))}$$

If $\eta(x)=1,\ \eta(x+1)=0$ $\quad p = (1-p)\,e^{\beta F}$

so "detailed balance" if

$$e^{\beta F} = \frac{p}{1-p}$$

($p=\frac{1}{2}$ is SEP and $F=0$).

Thus ASEP has invariant measure

$$\frac{1}{Z}\prod_x e^{\beta F x\eta(x)} = \prod_x \frac{e^{\beta F x\eta(x)}}{1+e^{\beta x F}}$$

i.e. $\prod_x \nu^0_{\rho(x)}$ product of Bernoulli with

density $$\rho(x) = \frac{e^{\beta F x}}{1+e^{\beta F x}}$$

For $F>0$, $\rho(x) \sim 1 - e^{-\beta F x} \xrightarrow[x\to\infty]{} 1$

$$\sim e^{\beta F x} \xrightarrow[x\to-\infty]{} 0$$

i.e. density