Gradient condition

consider symmetric exclusion.

Current $j(x,y,\eta) = c(x,y,\eta)(\eta(x)-\eta(y)) =$

$p(x,y)(\eta(x)(1-\eta(y)) + \eta(y)(1-\eta(x)))(\eta(x)-\eta(y))$

$= p(x,y)(\eta(x)-\eta(y))$ (since $\eta(x) \neq \eta(y)$)

So $j(\eta) = \sum_y (v\cdot y)\, p(0,y)\,(\eta(0)-\eta(y))$

and $\sum \tau_x j = \sum_y (v\cdot y)\, p(0,y) \sum_x (\eta(x)-\eta(x-y))$

$= 0$

Thus current contribution is and

$$D_{\alpha\beta} = \frac{1}{4\chi} \sum_x x_\alpha x_\beta \, E(c(0,x,\eta))$$

$$= \frac{1}{4\chi} \sum_x x_\alpha x_\beta \, p(0,x) \cdot 2(\rho(1-\rho))$$

$$\chi = \sum_x E(\eta(0)\eta(x)) - \rho^2 = E\,\eta(0)^2 - \rho^2 = \rho(1-\rho)$$

so

$$\underline{D_{\alpha\beta} = \frac{1}{2} \sum x_\alpha x_\beta \, p(0,x)}$$

Def The rates $c(x,y,\eta)$ satisfy the *gradient condition* if the current

$$j(x,y,\eta) \equiv c(x,y,\eta)(\eta(x)-\eta(y)) = k(x-y)(\tau_x h(\eta) - \tau_y h(\eta))$$

for some local function $h(\eta)$

Example For symmetric (transl inv) exclusion $k(x-y) = p(x,y)$, $h(\eta) = \eta(0)$

If gradient condition holds, then

$$j(\eta) = \sum_y (v \cdot y)\, r(y)\, (\tau_y h - h)$$

$$\sum_x \tau_x j = \sum_y (v \cdot y)\, r(y) \sum_x (\tau_{x+y} h - \tau_x h) = 0$$

Thus current doesn't contribute and

$$D_{\alpha\beta} = \frac{1}{4\chi} \sum x_\alpha x_\beta \, E(c(0,x,\eta))$$

is explicitely given.

For typical rates gradient condition does not hold. A class where it does is

<u>Zero-range process</u> : $c(x,y,\eta) = p(x,y) f(\eta(x))$

where now $\eta(x) \in \mathbb{N}$. Jump rate depends only on the particle # at x, i.e. range of interaction is zero. Let $p(x,y) = p(y,x) = r(x-y)$. Then

$$j(x,y,\eta) = c(x,y,\eta) - c(y,x,\eta) = r(x-y)\,(f(\eta(x)) - f(\eta(y)))$$

so satisfies gradient condition.

Hydrodynamic limit

consider reversible lattice gas in box $\mathbb{T}_N^d \equiv \Lambda_N$

$$Lf = \sum_{x,y \in \Lambda_N} c(x,y,\eta)\,(f(\eta^{x,y}) - f(\eta))$$

Reversible measure $\mu^N(\eta) = \frac{1}{Z}\, e^{[-\beta H + \lambda \sum_{x \in \Lambda_N} \eta(x)]}$

This has uniform density

$$\rho = E^{\mu^N}(\eta(x))$$

At $\beta = 0$, $\rho = \sum_{\eta = 0,1} \eta e^{\lambda\eta} / \sum e^{\lambda\eta} = \frac{e^{\lambda}}{1 + e^{\lambda}}$

Of course $S(t)\mu = \mu$.

Take a measure close to μ but with

slowly varying density

Eg take λ x-dependent

$$\mu = \frac{1}{Z}\, e^{[-\beta H + \sum \eta(x)\tilde{\lambda}(x)]}$$

At $\beta = 0$,

$$\rho(x) = E\,\eta(x) = \frac{e^{\tilde{\lambda}(x)}}{1 + e^{\tilde{\lambda}(x)}}$$

Take $\tilde{\lambda}(x) = \lambda(x/N)$ $\quad \lambda \in C^\infty(\mathbb{T}^d)$

Call the initial measure μ_λ^N.

At $\beta = 0$ we get for $f \in \mathcal{C}$ (local f)

$$\int f \, d\mu_\lambda^N \xrightarrow[N \to \infty]{} \int f \, d\mu_{\lambda(0)}$$

where $\mu_{\lambda(0)}$ is Bernoulli on $\{0,1\}^{\mathbb{Z}^d}$ density $e^{\lambda(0)}/(1+e^{\lambda(0)})$ and

$$\lim_{N\to\infty} \int \tau_{[Nx]} f \, d\mu^N_\lambda = \int f \, d\mu_{\lambda(x)} \qquad (A)$$

Same holds for β small.

Thus local observable at $[Nx]$ sees $\mu_{\lambda(x)}$ as $N\to\infty$.

Let $\Lambda_M(y)$ be a cube of side M at $y \in \mathbb{T}^d_N$

Set $N, M \to \infty$, $\frac{M}{N} \to 0$. Then, in probability

$$\frac{1}{|\Lambda_M(Nx)|} \sum_{y\in\Lambda_M(Nx)} \eta(y) \to \rho(x)^{*)} \qquad (B)$$

where $\rho(x) = \int \eta(0) \, d\mu_{\lambda(x)}$.

This is Law of Large Numbers for independent variables (at $\beta=0$) and for weakly dependent ones (β small).

So macro averages become deterministic.

*) I.e. $\text{Prob}\left(\left|\frac{1}{\Lambda_M} \sum \eta(y) - \rho(x)\right| > \varepsilon\right) \to 0$ $\underline{\forall \varepsilon > 0}$

A family of measures μ^N satisfying (A) and B can be called a <u>Local Equilibrium</u> family.

Reformulate (B): Let $\varphi \in C^\infty(\mathbb{T}^d)$.

Set $n_N(\varphi) = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x)\, \varphi(x/N)$

Then: $\forall \varepsilon > 0$

$$\lim_{N\to\infty} P\left(\left| n_N(\varphi) - \int_{\mathbb{T}^d} \varrho(x)\varphi(x)\, dx \right| > \varepsilon \right) = 0$$

(C)

<u>Proof</u> Given integer K, let for $x \in \mathbb{R}^d$

$[x]_K = \frac{1}{K}[Kx]$, $[\cdot] =$ integer part.

We have:

$$\left| \varphi\left(\tfrac{x}{N}\right) - \varphi\left(\left[\tfrac{x}{N}\right]_K\right) \right| \le \|\nabla\varphi\|_\infty \cdot \frac{1}{K}$$

$$\left| \int_{\mathbb{T}^d} \varrho(x)\varphi(x)\, dx - \frac{1}{K^d} \sum_{y \in \mathbb{T}_K^d} \varrho\left(\tfrac{y}{K}\right)\varphi\left(\tfrac{y}{K}\right) \right|$$

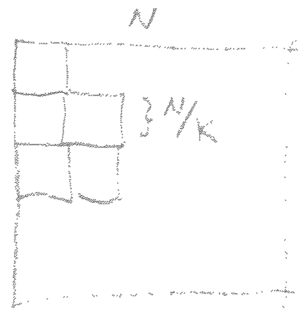
$$\le \|\nabla(\varrho\varphi)\|_\infty \frac{1}{K}$$

Pick K so big, both $\le \frac{\varepsilon}{4}$.

Then

$$\left| n_N(\varphi) - \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta(x)\, \varphi\left(\left[\tfrac{x}{N}\right]_K\right) \right| \le$$

$$\le \frac{1}{N^d} \sum_x \left| \varphi\left(\tfrac{x}{N}\right) - \varphi\left(\left[\tfrac{x}{N}\right]_K\right) \right| \le \varepsilon/4$$



and so

$$|n_N(\varphi) - \int \varrho\varphi| \le \frac{\varepsilon}{2} + \frac{1}{K^d} \sum_{y \in \mathbb{T}_K^d} \left| \varphi(y) \left(\left(\tfrac{K}{N}\right)^d \sum_{x \in \Lambda_{N/K}(y)} \eta(x) - \varrho(y) \right) \right|$$

But $\lim\limits_{N\to\infty} \mathbb{P}\left(\exists y \in \mathbb{T}_k^d : \left|\left(\frac{k}{N}\right)^d \sum \eta - \rho(y)\right| > \frac{\varepsilon}{2\|\varphi\|_\infty} \right)$

$= 0 \Rightarrow$ claim $\square$

Consider now time evolution. Let

$$n_N(t,\varphi) = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \varphi(x/N)\, \eta_t(x)$$

The hydrodynamical limit claim then is

$\forall \varphi,\ t \geq 0,\ \varepsilon > 0$:

$$\lim_{N\to\infty} \mathbb{P}\left(\left| n_N(N^2 t, \varphi) - \int_{\mathbb{T}^d} \varphi(x)\, \rho(t,x)\, dx \right| > \varepsilon \right) = 0$$

for $\rho(t,x)$ a solution of

$$\partial_t \rho = \sum_{\alpha\beta} \frac{\partial}{\partial x_\alpha}\left(K_{\alpha\beta}(\rho) \frac{\partial}{\partial x_\beta} \rho \right)$$

or more concisely: $\dot{\rho} = \nabla \cdot J(\rho)$

$J(\rho) = K(\rho) \nabla \rho$

$K : \mathbb{R} \to d \times d$ matrices.

For general rates this is not proven

We sketch a proof if the <u>gradient condition</u> holds.

Density field equation

We may write

$$n_N(t,\varphi) = \int_{\mathbb{T}^d} \varphi(y)\, n_N(t, dy)$$

where $n_N(t)$ is the measure

$$n_N(t) = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta_t(x)\, \delta_{x/N}$$

on $\mathbb{T}^d$. $n_N(t)$ is a random measure and we want to show

$$n_N(N^2 t) \xrightarrow[N\to\infty]{} \rho(t,x)\, dx \quad \text{in probability}$$

in some sense. We have

$$n_N(t,\varphi) = n_N(0,\varphi) + \int_0^t ds\, L\, n_N(s,\varphi) + M_N(t,\varphi)$$

where M_N is a martingale.

Recall

$$L\,\eta(x) = \frac{1}{2} \sum_{u y} c(u,y,\eta)\left(\eta^{uy}(x) - \eta(x)\right)$$

$$= \frac{1}{2} \sum_{y} c(x,y,\eta)\left(\eta(y) - \eta(x)\right) + \frac{1}{2} \sum_{u} c(u,x,\eta)\left(\eta(u) - \eta(x)\right)$$

$$= \sum_{y} c(x,y,\eta)\left(\eta(y) - \eta(x)\right)$$

$$= \sum_{y} j(x,y,\eta) \equiv \operatorname{div} j(x,\eta)$$

So

$$L\, n_N(s,\varphi) = \frac{1}{N^d} \sum_{x,y} j(x,y,\eta(s))\, \varphi(x/N)$$

Gradient condition:

$$j(x,y,\eta) = r(x-y)\left(\tau_x h(\eta) - \tau_y h(\eta)\right)$$

w

$$\underline{L\, n_N(s,\varphi)} = \frac{1}{N^d} \sum_{x,v} r(x-v)\left(\tau_x h(\eta) - \tau_v h(\eta)\right) \varphi(x/N)$$

$$= \frac{1}{N^d} \sum_{x} \tau_x h(\eta) \sum_{y} r(y) \left(\varphi\left(\frac{x}{N}\right) - \varphi\left(\frac{x+y}{N}\right) \right)$$

Here $r(y) = r(-y)$, $r(y) = 0$ $|y| > R$.

$$\underline{= -\frac{1}{N^d} \sum_{x} \tau_x h(\eta) (\tilde{\Delta}_N \varphi)\left(\frac{x}{N}\right)}$$

$$\tilde{\Delta}_N \varphi(z) = \frac{1}{2} \sum_{y} r(y) \left(\varphi\left(z + \frac{y}{N}\right) + \varphi\left(z - \frac{y}{N}\right) - 2\varphi(z) \right)$$

$$= \frac{1}{N^2} \sum_{\alpha\beta} D_{\alpha\beta}\, \partial_\alpha \partial_\beta \varphi + O \| D^3 \varphi \|_\infty \frac{1}{N^3}$$

$$D_{\alpha\beta} = \frac{1}{4} \sum_{y} y_\alpha y_\beta r(y)$$

Thus got (change variables $s \to N^2 s$)

$$n_N(N^2 t, \varphi) = n_N(0,\varphi) - \frac{1}{N^d} \sum_{x} \int_0^t (\tau_x h(\eta_{N^2 s}))(A\varphi)(x/N)$$
$$+ M_N(N^2 t, \varphi) + O(1/N)$$

$$A = \sum D_{\alpha\beta} \partial_\alpha \partial_\beta$$

<u>Martingale part</u>

Recall:

$$M_N(t,\varphi)^2 - \int_0^t ds \underbrace{\left(L\, n_N(s,\varphi)^2 - 2 n_N L n_N\right)}_{Q_N(t,\varphi)}$$

+ Martingale.

$$L\, n_N(s,\varphi)^2 = \frac{1}{N^{2d}} \sum_{x,y \in \mathbb{T}_N^d} \varphi\left(\frac{x}{N}\right) \varphi\left(\frac{y}{N}\right) f_{xy}(\eta(s))$$

with

$$f_{xy}(\eta) \equiv L(\eta(x)\eta(y))$$

$$= \frac{1}{2}\sum_{u,v} c(u,v,\eta)\,(\eta^{uv}(x)\eta^{uv}(y) - \eta(x)\eta(y))$$

$$2nLn = \frac{1}{N^{2d}}\sum_{x,y} \varphi(\tfrac{x}{N})\varphi(\tfrac{y}{N}))\; g_{xy}(\eta(s))$$

$$g_{xy}(\eta) = \eta(x)L\eta(y) + \eta(y)L\eta(x)$$ (symmetrized the 2

$$= \frac{1}{2}\sum_{uv} c(u,v,\eta)\,(\eta(x)(\eta^{uv}(y) - \eta(y))$$

$$+ \eta(y)(\eta^{uv}(x) - \eta(x))$$

So

$$f_{xy} - g_{xy} = \frac{1}{2}\sum_{uv} c(u,v,\eta)\,(\eta^{uv}(x) - \eta(x))(\eta^{uv}(y) - \eta(y))$$

This = 0 unless $\{x,y\} = \{u,v\}$ or $x = y = u$ or
$x = y = v$

$$= -c(x,y,\eta)(\eta(x) - \eta(y))^2 + \delta_{xy}\sum_{v} c(x,v,\eta)\,(\eta(x) - \eta(v))^2$$

Thus

$$\frac{1}{N^{2d}}\sum_{x,y} \varphi(\tfrac{x}{N})\varphi(\tfrac{y}{N}))\,(f_{xy} - g_{xy})$$

$$= \frac{1}{N^{2d}}\sum_{x,v} \varphi(\tfrac{x}{N})^2\, c(x,v,\eta)(\eta(x) - \eta(v))^2 - \sum_{x,y} \varphi(\tfrac{x}{N})\varphi(\tfrac{y}{N})\,c(x,y,\eta)\,(\eta(x) - \eta(y))^2$$

$$= \frac{1}{N^{2d}}\sum_{x,y} \varphi(\tfrac{x}{N})(\varphi(\tfrac{x}{N}) - \varphi(\tfrac{y}{N}))\,c(x,y,\eta)(\eta(x) - \eta(y))^2$$

$$= \frac{1}{2N^{2d}}\sum_{x,y} (\varphi(\tfrac{x}{N}) - \varphi(\tfrac{y}{N}))^2\, c(x,y,\eta)(\eta(x) - \eta(y))^2$$

$$< \|\nabla\varphi\|_\infty^2 \cdot C \cdot \frac{1}{N^2}\,\frac{1}{N^d}$$

Thus we get

$$Q_N(N^2 t, \varphi) = \int_0^t ds \; \gamma_N(\eta_{N^2 s})$$

with

$$0 \le \gamma_N(\eta) \le \frac{c}{N^d} \|\nabla \varphi\|_\infty^2$$

Doob-inequality says:

$$E(\sup_{s \le t} |M(s)|)^2 \le 4 \, E\, M(t)^2$$

for a martingale M. Hence, since $E M^2 = E Q$

get

$$E(\sup_{s \le t} |M(N^2 s, \varphi|)|)^2 \le \frac{c\,t}{N^d} \|\nabla \varphi\|^2$$

Thus, by Chebyshev

$$\lim_{N \to \infty} P\Big(\sup_{\tau \le t} | n_N(N^2 \tau, \varphi) - n_N(0, \varphi) + \int_0^\tau \frac{1}{N^d} \sum_x \tau_x h(\eta_{N^2 s})) A\varphi(\tfrac{x}{N}) ds | > \varepsilon \Big) = 0 \qquad (*)$$

$\forall \varepsilon > 0$

Ex. Symmetric exclusion process

Recall that in this case $h(\eta) = \eta(0)$ so $(*)$ reads

$$\lim_{N \to \infty} P(\sup | n_N(N^2 \tau, \varphi) - n_N(0, \varphi) + \int_0^\tau n_N(N^2 s, A\varphi) | > \varepsilon) = 0$$

i.e. only n_N enters here!

We now want to conclude that $\eta_N(N^2 s)$ converges in some sense.

1. $\eta_N(N^2 s) \in \mathcal{M}(\mathbb{T}^d) \equiv$ positive Borel measures on $\mathbb{T}^d$ with finite total mass

This is a Polish space, with Prokhorov metric

Topology of weak convergence.

$$\nu_n \to \nu \quad \text{if} \quad \int \nu_n f \to \int \nu f \quad \forall f \in C(\mathbb{T}^d)$$

2. $(\eta_N(N^2 s))_{s \in [0,T]} \in D([0,T], \mathcal{M}(\mathbb{T}^d))$

$=$ LCRL functions $[0,T] \longrightarrow \mathcal{M}(\mathbb{T}^d)$

This has Skorokhod metric.

3. The measure P^{μ^N} on $D([0,T], \{0,1\}^{\mathbb{T}_N^d})$ of the Markov process η_t with initial measure μ^N induces a measure P_N on $D([0,T], \mathcal{M}(\mathbb{T}^d))$, the distribution of $\eta_N(s)$.

4. Show: $P_N \xrightarrow[N \to \infty]{} \delta_u$

where δ_u is the Dirac measure on the deterministic path

$$\begin{cases} \dot{u} = Au \\ u(0,x) = g(x) \end{cases} \qquad A = \sum D_{\alpha\beta} \partial_\alpha \partial_\beta$$

5. To show P^N converges, show

a) P^N has a convergent subsequence

b) All limit points are measures concentrated on <u>weak solutions</u> of 4 i.e $u(t,x) \in L^\infty$ s.t. $\forall \varphi \in C^\infty(\mathbb{T}^d)$

$$\int u(t,x)\varphi(x)dx - \int u(0,x)\varphi(x)dx =$$
$$= \int_0^t\int u(s,x)(A\varphi)(s,x))ds\,dx$$

c) Show weak solutions are unique and are classical solution u.

6. To show 5a need to show (P^N) is relatively compact. For this we apply Prohorov theorem.

<u>Prohorov thm</u> Let $\mu_n \in \mathcal{M}_1(Y)$ Y Polish space. Suppose $(\mu_n)_{n\geq 1}^\infty$ is <u>Tight</u> i.e. $\forall \varepsilon > 0$ $\exists$ compact $K_\varepsilon \subset Y$ s.t

$$\mu_n(K_\varepsilon) \geq 1-\varepsilon \quad \forall n$$

Then (μ_n) is relatively compact i.e $\exists$ subsequence μ_{n_i} that converges.

8. If one works out what tightness means in our context it turns out it is sufficient to show that for each $\varphi \in C^\infty(\mathbb{T}^d)$, the family of prob. measures P^N_ϕ on $D([0,T], \mathbb{R})$ are tight; P^N_ϕ is the distribution of $n_N(\cdot, \phi)$

$$P^N_\phi(A) = P^N(\, n_N(\cdot, \phi) \in A\,)$$

To show P^N_ϕ is tight need: a) $n_N(t,\varphi)$ is in compact set of $\mathbb{R}$ w. high probability fixed t. b) $n_N(t,\varphi)$ have an equicontinuity property. These follow from our martingale estimates. See eg. Seppäläinen

9. Next one shows any limit point is a measure $n(t) = n(t,x)\,dx$ $n(t,x) \in L^\infty$.

This is, because $|n_N(t,\varphi)| \le \frac{1}{N^d}\sum |\varphi(x/N)|$

This carries to the limit point: $|n(t,\varphi)| \le \int_{\mathbb{T}^d} |\varphi|\,d\cdot$ $\forall \varphi$ so the claim follows. One then also gets $n(t,x)$ satisfies

$$\int n(t,x)\varphi(x)\,dx - \int g(x)\varphi(x)\,dx =$$

$$\int_0^t ds \int_{\mathbb{T}^d} n(s,x)\,\Delta\varphi(x)\,dx$$

ie n is a <u>weak solution</u> to $\begin{cases} \dot{n} = \Delta n \\ n(0,x) = g(x) \end{cases}$

General (Gradient) Case

In general we have obtained

$$\lim_{N\to\infty} P\Big(\sup_{t'\le t} \Big| n_N(N^2 t', \varphi) - n_N(0,\varphi) + \frac{1}{N^d} \sum_x \int_0^{t'} (\tau_x h)(\eta_{N^2 s})(A\varphi)\big(\tfrac{x}{N}\big) \Big|$$

$$> \varepsilon \Big) \quad = 0 \qquad (1)$$

Let μ_ρ be the equilibrium state with density ρ.

We take $\beta=0$ so μ_ρ is Bernoulli, $P(\eta=1)=\rho$.

Let
$$\tilde{h}(\rho) = E^{\mu_\rho} h = \int h(\eta)\, d\mu_\rho(\eta)$$

Our aim is to close eq (1) i.e. write it in terms of n_N only. The limiting equation will be for $u(t,x)\,dx = \lim_N n_N(N^2 t, dx)$

$$\int (u(t,x) - u(0,x))\,\varphi(x)\,dx = \int_0^t ds \int dx\ \tilde{h}(u(s,x))(A\varphi)(x)$$

or : $\begin{cases} \dot{u} = A\, h(u) \\ u(0,x) = \rho(x) \end{cases}$ $\qquad A = \sum D_{\alpha\beta} \frac{\partial}{\partial x_\alpha} \frac{\partial}{\partial x_\beta}$

For $\Lambda \subset \mathbb{T}_N^d$ a cube, let $\langle - \rangle_\Lambda$ denote spatial average : so

$$\langle h \rangle_\Lambda = \frac{1}{|\Lambda|} \sum_{x\in\Lambda} (\tau_x h)(\eta)$$

$$\langle \eta \rangle_\Lambda = \frac{1}{\Lambda} \sum_{x\in\Lambda} \eta(x)$$

Let $\mu_{N,t} = S(t)\mu_N$

where μ_N is the Bernoulli $E\eta(x) = \rho(x/N)$.

Denote by $\Lambda(x, M) \subset \mathbb{T}_N^d$ a cube with

center x, side M.

At time $t=0$ we have

$$\langle h \rangle_{\Lambda(Nx,M)} \approx \tilde{h}\left(\langle \eta \rangle_{\Lambda(Nx,M)}\right)$$

with high μ_N probability i.e.

$$\frac{1}{|\Lambda|} \sum_{y \in \Lambda} (\tau_y h)(\eta) \approx E^{\mu_\rho} h \Big|_{\rho = \frac{1}{|\Lambda|} \sum_{y\in\Lambda} \eta(y)}$$

More precisely let $N, M \to \infty$, $\frac{M}{N} \to 0$;

$$\lim_{N\to\infty} P_{\mu_N}\left(\left| \langle h \rangle_{\Lambda(Nx,M)} - \tilde{h}\left(\langle \eta \rangle_{\Lambda(Nx,M)}\right) \right| > \varepsilon \right) = 0 \quad \forall \varepsilon > 0$$

We prove similar claim at time t $\forall \varphi$:

Prop

$$\lim_{\ell\to 0} \lim_{N\to\infty} \int_0^t ds\, E_N \left(\left| \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \varphi\left(\frac{x}{N}\right) \left[(\tau_x h)(\eta_{N^2 s}) - \tilde{h}\left(\langle \eta_{N^2 s} \rangle_{\Lambda(Nx,\ell N)} \right) \right] \right| \right) = 0 \qquad (2$$

where E_N is the path space measure corresponding with μ_N initial condition.

This result is consistent with the claim that $\mu_{N,N^2 s}$ is close to a local equilibrium.

measure. The latter claim is hard to prove directly. Note that (2) can be written in terms of the empirical measure π_N:

$$\langle \eta_{N^2 s} \rangle_{\Lambda_{\ell N}(Nx)} = \frac{1}{|\Lambda_{\ell N}(Nx)|} \sum_{y \in \Lambda_{\ell N}(Nx)} \eta_{N^2 s}(y)$$

$$= \pi_N(N^2 s, \delta_x^\ell)$$

where $\delta_x^\ell(y) = \ell^{-d} \cdot \begin{cases} 1 & y \in \Lambda_\ell(x) \\ 0 & \text{otherwise} \end{cases}$

(so $\delta_x^\ell \xrightarrow[\ell \to 0]{} \delta(x-y)$)

Thus, defining

$$\tilde{h}_\ell(\pi_N, x) \equiv \tilde{h}(\pi_N(\delta_x^\ell))$$

we got (2) $\Leftrightarrow$

$$\lim_{\ell \to 0} \lim_{N \to \infty} \int_0^t ds \; E_N \left| \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} g\left(\frac{x}{N}\right) (\tau_x h)(\eta_{N^2 s}) - \tilde{h}_\ell(\pi_N, x) \right| = 0$$

Proof of Proposition

1. Time averages

Let

$$\bar{\mu}_N = \frac{1}{t} \int_0^t ds \; \mu_{N,s} \qquad (\mu_{N,s} = S(s)\mu_N)$$

be the time-average of the measures $\mu_{N,s}$.

We have:

$$\int_0^t ds\; E_N\Big(\Big|\frac{1}{N^d}\sum\varphi(\tfrac{x}{N})(\tau_x h)(\eta_{N^2 s}) - \tilde h(\langle\eta_{N^2 s}\rangle_{\Lambda_{\ell N}(Nx)}\Big)\Big|$$

$$= t\;\bar E_N\Big(\Big|\frac{1}{N^d}\sum\varphi(\tfrac{x}{N})(\tau_x h)(\eta) - \tilde h(\langle\eta\rangle_{\Lambda_{\ell N}(Nx)})\Big|\Big)$$

$$= t\;\bar E_N\Big(\frac{1}{N^d}\sum_x\varphi(\tfrac{x}{N})\Big[\langle h\rangle_{\Lambda_{\ell N}(Nx)}(\eta) - \tilde h(\langle\eta\rangle_{\Lambda_{\ell N}(Nx)})\Big]\Big|\Big)$$

$$+\, t\;\bar E_N\Big(\Big|\frac{1}{N^d}\sum_x\varphi(\tfrac{x}{N})\big(\tau_x h - \langle h\rangle_{\Lambda_{\ell N}(Nx)}\big)\Big|\Big)$$

$$\leq t\,\|\varphi\|_\infty\;\bar E_N\Big(\frac{1}{N^d}\sum_x\Big|\langle h\rangle_{\Lambda_{\ell N}(Nx)} - \tilde h(\langle\eta_{\Lambda_{\ell N}(Nx)}\rangle\Big|\Big)$$

$$+\, t\,\|h\|_\infty\underbrace{\frac{1}{N^d}\sum_x\Big|\varphi(\tfrac{x}{N}) - \frac{1}{(\ell N)^d}\sum_{y\in\Lambda_{\ell N}(Nx)}\varphi(y)\Big|}_{\leq\;\ell\,\|\nabla\varphi\|_\infty\;\xrightarrow[\ell\to 0]{}\;0}$$

Thus need to show

$$\lim_{\ell\to 0}\lim_{N\to\infty}\bar E_N\Big(\frac{1}{N^d}\sum_x\Big|\langle h\rangle_{\Lambda_{\ell N}(x)} - \underbrace{\tilde h(\langle\eta\rangle_{\Lambda_{\ell N}})}_{=E_{\nu\langle\eta\rangle}h}\Big|\Big) = 0 \qquad (3)$$

2. Entropy production

We control time dependence of $\mu_{N,t}$ using entropy.

Def Let ν = Bernoulli-$1/2$ measure on $\{0,1\}^{\mathbb{T}_N^d}$ i.e. $E_\nu\,\eta(x) = 1/2$. Let

$\mu = f\nu$, $\int f\nu = 1$. Then

Entropy of μ .

$$F(\mu) = E_\nu (f \log f)$$

(where if $f(\eta)=0$ set $f(\eta) \log f(\eta) \equiv 0$).

<u>Homework</u> a) $0 \leq F(\mu) \leq N^d \log 2$

& $F(\mu)=0 \Leftrightarrow f=1$ (i.e. $\mu=\nu$)

<u>Def</u> Entropy production.

$$\sigma(\mu_{N,t}) \equiv -\frac{d}{dt} F(\mu_{N,t}) = -\frac{d}{dt} F(S^*(t)\mu_N)$$

$$= -\frac{d}{dt} F(S(t) f \nu)$$

$$= -\frac{d}{dt} E_\nu (e^{tL} f \log e^{tL} f)$$

$$= - E_\nu (L f_t \log f_t + L f_t) \qquad f_t = S(t) f$$

$$\underset{\int Lf \nu = 0}{=} - E_\nu (L f_t \log f_t)$$

$$= -\frac{1}{2} \sum_{x,y} E\Big(c(x,y,\eta) (f_t(\eta^{x,y}) - f_t(\eta)) \log f_t(\eta) \Big)$$

$$= \frac{1}{4} \sum_{x,y} E \Big(c(x,y,\eta) (f_t(\eta^{x,y}) - f_t(\eta)) (\log f_t(\eta^{x,y}) - \log f_t(\eta)) \Big)$$

$$\equiv \sigma (f_t)$$

So we have

$$F(\mu_{N,N^2t}) = F(\mu_N) - N^2 \int_0^t ds\, \sigma(\mu_{N,s})$$

Since $0 \leq F \leq N^d \log 2$ get

$$\frac{1}{t} \int_0^t ds\, \sigma(\mu_{N,s}) \leq \frac{1}{t} N^{d-2} \log 2$$

Homework σ is convex

Hence

$$\sigma(\bar{\mu}_N) \leq \frac{1}{t}\int_0^t ds\, \sigma(\mu_{N,s}) \leq \frac{1}{t} N^{d-2} \log 2$$

This trivial inequality is all we use about the time evolution!

Consider (3). The expectation is of a translation invariant observable so we may replace $\bar{\mu}_N$ by its spatial average

$$\tilde{\mu}_N = \frac{1}{N^d}\sum_{x\in \mathbb{T}_N^d} \tau_x \mu_N$$

i.e.

$$\tilde{\mu}_N = \tilde{S}\,\nu$$

$$\tilde{S} = \frac{1}{N^d}\sum_{x\in \mathbb{T}_N^d} \frac{1}{t}\int_0^t ds\quad \tau_x\, S(s)\, f$$

i.e. our claim follows from

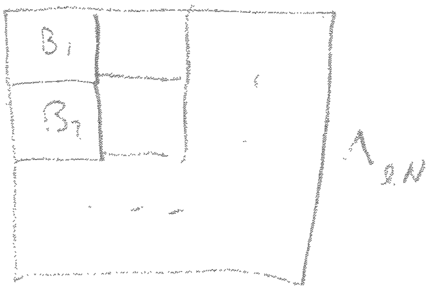
Prop

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty}\sup_{f\in A_N}\int \left|\langle h\rangle_{\Lambda_{\varepsilon N}(0)} - \tilde{h}(\langle\eta\rangle_{\Lambda_{\varepsilon N}(0)})\right| f\, d\nu = 0 \qquad (*)$$

where

$$A_N = \{ f \mid \sigma(f) \leq \frac{1}{t} N^{d-2}\log 2,\ f \geq 0,\ \int f\, d\nu = 1,$$
$$\text{translation invariant}\}$$

Proof



Decompose $\Lambda \equiv \Lambda_{\ell N} = \bigcup_{i=1}^{K} B_i$ $\quad B_i$ M-cube

So
$$|\langle h\rangle_\Lambda - \tilde{h}(\langle\eta\rangle_\Lambda)| \le \Big|\frac{1}{K}\sum_{i=1}^{K}\langle h\rangle_{B_i} - \tilde{h}(\langle\eta\rangle_\Lambda)\Big|$$
$$\le \frac{1}{K}\sum_{i=1}^{K}\Big(|\langle h\rangle_{B_i} - \tilde{h}(\langle\eta\rangle_{B_i})| + |\tilde{h}(\langle\eta\rangle_{B_i}) - \tilde{h}(\langle\eta\rangle_\Lambda)|\Big)$$

We estimate

$$|\tilde{h}(\langle\eta\rangle_{B_i}) - \tilde{h}(\langle\eta\rangle_\Lambda)| \le \|\tilde{h}'\|_\infty\, |\langle\eta\rangle_{B_i} - \langle\eta\rangle_\Lambda|$$
$$\le \|\tilde{h}'\|_\infty \frac{1}{K}\sum_{j=1}^{K} |\langle\eta\rangle_{B_i} - \langle\eta\rangle_{B_j}|$$

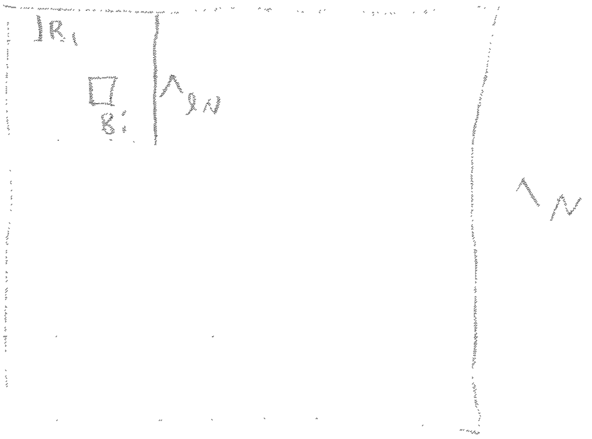
We will prove:

$$\lim_{M\to\infty}\lim_{N\to\infty}\sup_{f\in A_N}\int |\langle h\rangle_{B_1} - \tilde{h}(\langle\eta\rangle_{B_1})|\, f\, d\nu = 0 \qquad (A)$$

"one-block estimate"

$$\lim_{M\to\infty}\lim_{\ell\to 0}\limsup_{N\to\infty}\sup_{1\le i\le K}\sup_{f\in A_N}\int |\langle\eta\rangle_{B_1} - \langle\eta\rangle_{B_i}|\, f\, d\nu = 0 \qquad (B)$$

'two-blocks estimate'



These imply the proposition (by translation invariance can replace B_1 by B_i).

(*): $\langle h \rangle_\Lambda \approx \tilde{h}(\langle \eta \rangle_\Lambda)$ $\Lambda = \Lambda_{\ell N}$ small macroscopic box $|\Lambda| = (\ell N)^d$

(A): $\langle h \rangle_B \approx \hat{h}(\langle \eta \rangle_B)$ B large microscopic box $|B| = M^d$

(B) $\langle \eta \rangle_{B_i} \approx \langle \eta \rangle_{B_j}$ B_i large microscopic, near macroscopically $\operatorname{dist}(B_i, B_j) \leq \ell N$, $\operatorname{dist}(\frac{B_i}{N}, \frac{B_j}{N}) \leq \ell \to 0$

We will replace entropy by <u>Dirichlet form</u>.

Define $Q(g) = \frac{1}{2} \sum_{uv} E_\nu(c(u,v,\eta)(g(\eta^{uv}) - g(\eta))^2$

<u>Lemma</u> $Q(\sqrt{f}) \leq \sigma(f)$

<u>Pf</u> use $2(\sqrt{u} - \sqrt{v})^2 \leq (u-v)(\log u - \log v)$ □

Suppose as always: $c(u,v,\eta) \geq c > 0$ if $|u-v| = 1$.

Then

$$Q(g) \geqslant c\, D(g) := c \sum_{|u-v|=1} \int (g(\eta^{uv}) - g(\eta))^2 d\nu$$

and so $c D(\sqrt{f}) \leq \sigma(f)$.

<u>Thus we may replace in A_N $\sigma(f)$ by $cD(\sqrt{f})$</u>

<u>Subadditivity</u> Let $B \subset \Lambda_N$ be a cube

and let f_B, f_{B^c} be the marginals

of f in $\{0,1\}^B$, $\{0,1\}^{B^c}$ i.e.

$$f_B(\eta) = \int f(\eta, \eta^c)\, \nu(d\eta^c) = \sum_{\eta^c \in \{0,1\}^{B^c}} f(\eta, \eta^c) (\tfrac{1}{2})^{|B^c|}$$

where $\eta \in \{0,1\}^B$, $\eta^c \in \{0,1\}^{B^c}$.

Define $D_B(f_B)$ with $\sum_{u,v \in B}$ and $D_{B^c}(f_{B^c})$ similarly.

Lemma $D(\sqrt{f}) \geq D_B(\sqrt{f_B}) + D_{B^c}(\sqrt{f_{B^c}})$

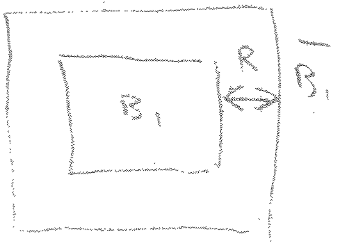
Proof Homework $\square$

Now, consider (A):

$$\int [\langle h \rangle_{B_1} - \tilde{h}(\langle \eta \rangle_{B_1})] f \, d\nu$$

$$= \int [\langle h \rangle_{B_1} - \tilde{h}(\langle \eta \rangle_{B_1})] f_{\bar{B}_1} \, d\nu_{\bar{B}_1}$$

where $\bar{B}_1$ = cube



R range of h

i.e. $h(\eta)$ depends

on $\eta(x), x \in \Lambda_R(0)$. I.e. $\bar{B}_1 = \Lambda_{M+2R}(0)$.

Write $$\mathbb{T}_N^d = \bigcup_{i=1}^{L} C_i \qquad C_i \quad M+R\text{-cubes}$$

$$L = \left[\frac{N}{M+2R}\right]^d$$

By Lemma

$$D(\sqrt{f}) \geq \sum_i D_{C_i}(f_{C_i})$$

By translation invariance $D_{C_i}(f_{C_i}) = D_{\bar{B}_1}(f_{\bar{B}_1})$

so

$$D_{\bar{B}_1}(f^{1/2}_{\bar{B}_1}) \leq \left(\frac{M+2R}{N}\right)^d D(f^{1/2})$$

$$\leq \frac{1}{2} (M+2R)^d N^{-2} \log 2$$

Thus, as $N \to \infty$, for the 1-block estimate need to study

$$\sup_{f} \int \underbrace{|\langle h\rangle_{B_1} - \tilde{h}(\langle \eta\rangle_{B_1})|}_{\equiv F} f \, d\nu_{\bar{B}_1} \qquad (*)$$

where sup is over $f : \{0,1\}^{\bar{B}_1} \to \mathbb{R}^+$

with $\int f \, d\nu_{\bar{B}_1} = 1$ and $D_{\bar{B}_1}(f^{1/2}) = 0$

i.e.

$$f^{1/2}(\eta^{x,y}) = f^{1/2}(\eta) \quad \forall x, y$$

This is solved by $f^{1/2}(\eta) = g(\sum_{x \in \bar{B}_1} \eta_x)$ (Homework)

i.e. $f(\eta) = h(\sum \eta_x)$.

Thus,
$$(*) = \sum_{n=0}^{P} \mathbb{E}_{\mu_n}(F) \, h(n) \qquad (**)$$

where $P = (M+2R)^d$, $\mu_n = \nu_{\bar{B}_1}$ conditioned on the event $\sum_{x \in \bar{B}_1} \eta(x) = n$. We have also

$$\sum_{n=0}^{P} h(n) = \mathbb{E}_{\nu_{\bar{B}}}(1) = 1.$$

We show: $\max_n E_{\mu_n}(F) \xrightarrow[M\to\infty]{} 0$. This $\Rightarrow$ claim.

Note $\langle \eta\rangle_{\bar{B}_1} = \frac{n}{P}$ on μ_n so since

$|\langle \eta\rangle_{B_1} - \langle \eta\rangle_{\bar{B}_1}| \leq C/M$ enough to show

$$\max_n \int |\langle h\rangle_{B_1} - \tilde{h}(\tfrac{n}{P})| \, d\mu_n \to 0 \quad M\to\infty$$

Here $\tilde{h}(\frac{n}{p}) = \mathbb{E}_{\nu_{n/p}} h$, $\nu_{n/p}$ Bernoulli density n/p.

<u>Homework</u> $\mathbb{E}\left(\langle h \rangle_{B_1} - E_{\mu_n} h\right)^2 \leq \frac{c}{m^d}$

Hence

$$\text{Prob}_{\mu_n}\left(|\langle h \rangle_{B_1} - E_{\mu_n} h| > A\right) \leq \frac{c}{A^2 m^d}$$

Hence suffices to show (Homework)

$$\max_n \int |\mathbb{E}_{\mu_n} h - E_{\nu_{n/p}} h| \, d\mu_n \to 0$$

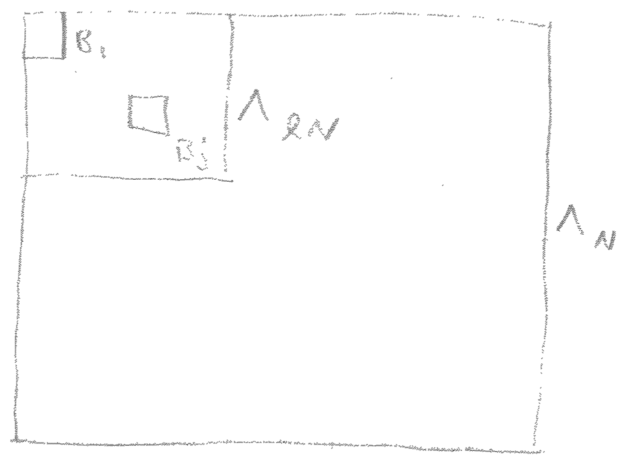
as $m \to \infty$

[Here $\mathbb{E}_{\nu_{n/p}}$ is in canonical ensemble: $\mathbb{E}_{\nu_{n/p}} \eta = n/p$, $\mathbb{E}_{\mu_n}$ is in microcanonical ensemble: $\frac{1}{p} \sum \eta = n/p$]

Thus $(**) \to 0$ as $m \to \infty$ and 1-block proven $\square$

2. Blocks estimate (Proof)

Recall



$\ell << 1$

$\Lambda_{\ell N} = \bigcup_{i=1}^{K} B_i$

$B_i = M$ - cube

Prop $\lim_{M\to\infty} \lim_{\ell\to 0} \lim_{N\to\infty} \sup_{i\le K} \sup_{f\in A_N} \int |\langle\eta\rangle_{B_1} - \langle\eta\rangle_{B_i}| f \, d\nu$

$A_N = \{ f \mid \int f \, d\nu = 1, \; D(\sqrt{f}) \le N^{d-2} \ell^{-1} \log 2, \; f$ translation invariant$\}$

Pf. We have by Schwartz ($f \, d\nu$ is prob. measure)

$$\left(\int |\langle\eta\rangle_{B_1} - \langle\eta\rangle_{B_i}| f \, d\nu \right)^2 \le \int (\langle\eta\rangle_{B_1} - \langle\eta\rangle_{B_i})^2 f \, d\nu$$

Remark This equals

$$M^{-2d} \Big[\sum_{x,y\in B_1} + \sum_{x,y\in B_i} - 2 \sum_{x\in B_1, y\in B_i} \Big] \int \eta(x)\eta(y) f \, d\nu \quad (*)$$

Suppose $f(\eta) = f(\eta^{x,y})$ $\forall x \in B_1, y \in B_i$.

Then $\int \eta(x)\eta(y) f \, d\nu = \int \eta(u)\eta(v) f \, d\nu$

if $x, y \in B_1$, $u, v \in B_i$ or $x \in B_1$, $y \in B_i$, $u \in B_1$, $v \in B_i$ and $x \ne y$, $u \ne v$. So $(*) = O(M^{-d})$.

We prove

<u>Lemma</u>

$$\int |\langle \eta \rangle_{B_1} - \langle \eta_{B_2} \rangle|^2 f \, d\nu$$

$$\leq C [M^{-d} + D_{11}(\sqrt{f})]$$

$$D_{11}(g) = \frac{1}{M^{2d}} \sum_{\substack{x \in B_1 \\ y \in B_1}} (g(\eta^{xy}) - g(\eta))^2$$

<u>Pf</u> <u>1st note</u>: for $w(\eta) > 0 \ \forall \eta$,

$$- \int w^{-1} A w f \, d\nu \leq D_{11}(\sqrt{f}) \qquad (1)$$

where

$$A w = 2 M^{-2d} \sum_{\substack{x \in B_1 \\ y \in B_1}} w(\eta^{xy}) - w(\eta)$$

Indeed,

$$- w^{-1} A w = 2 M^{-2d} \sum_{x,y} \left(1 - \frac{w(\eta^{xy})}{w(\eta)} \right)$$

and, for $g = \sqrt{f}$

$$2 \int f \left(1 - \frac{w(\eta^{xy})}{w(\eta)}\right) d\nu = \int \Big[g(\eta) g(\eta) \left(1 - \frac{w(\eta^{xy})}{w(\eta)}\right)$$

$$+ g(\eta^{xy}) g(\eta^{xy}) \left(1 - \frac{w(\eta)}{w(\eta^{xy})}\right) \Big] d\nu$$

$$= (g(\eta^{xy}) - g(\eta))^2 - \left(2 g(\eta^{xy}) g(\eta) + g g \frac{w(\eta^{xy})}{w} + g^{xy} g^{xy} \frac{w}{w^{xy}}\right)$$

$$= (g(\eta^{xy}) - g(\eta))^2 - \left(g(\eta) \left(\frac{w(\eta^{xy})}{w(\eta)}\right)^{1/2} + g(\eta^{xy}) \left(\frac{w(\eta)}{w(\eta^{xy})}\right)^{1/2} \right)^2$$

$$\leq (g(\eta^{xy}) - g(\eta))^2 \Rightarrow (1) \ \square$$

Now take

$$w = \exp\left[\frac{1}{2} M^{d} \left(\langle \eta \rangle_{B_1}^2 + \langle \eta \rangle_{B_i}^2 \right)\right]$$

$$= \exp\left[\frac{1}{2} M^{-d} \Big(\sum_{x,y \in B_1} + \sum_{x,y \in B_i} \Big) \eta(x)\eta(y) \right] = e^F$$

let $u \in B_1$, $v \in B_i$, $\partial_{uv} g = g(\eta^{uv}) - g(\eta)$

$$e^{-F} \partial_{uv} w = \left(e^{F(\eta^{uv})} - e^{F(\eta)}\right) e^{-F}$$

$$= \left(e^{\partial_{uv} F} - 1\right)$$

$$\partial_{uv} F = \frac{1}{2} M^{-d} \Big[(\eta(v)-\eta(u)) \sum_{\substack{y \neq u \\ y \in B_1}} \eta(y) + (\eta(u)-\eta(v)) \sum_{\substack{y \neq v \\ y \in B_i}} \eta(y) \Big]$$

$$\frac{1}{2} (\eta(v)-\eta(u)) \left(\langle \eta \rangle_{B_1} - \langle \eta \rangle_{B_i} \right) + \frac{1}{2} M^{-d} \left[-(\eta(v)-\eta(u))\eta(u) - (\eta(u)-\eta(v))\eta(v) \right]$$

We have

$$|\partial_{uv} F| \leq \frac{1}{2} + O(M^{-d})$$

so using $1 - e^x \geq -cx \quad |x| \leq 1$

$$-w^{-1} \partial_{uv} w = 1 - e^{\partial_{uv} F} \geq -c\, \partial_{uv} F$$

$$-\frac{1}{M^{2d}} \sum_{u \in B_1} \sum_{v \in B_i} w^{-1} \partial_{uv} w \geq -2c \left[\left(\langle \eta \rangle_{B_i} - \langle \eta \rangle_{B_1} \right) \left(\langle \eta \rangle_{B_1} - \langle \eta \rangle_{B_i} \right) \right]$$

$$- O(M^{d})$$

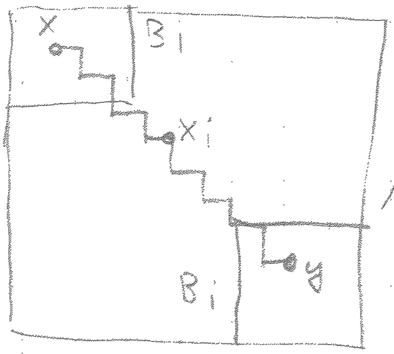
$$= 2c \left(\langle \eta \rangle_{B_1} - \langle \eta_{B_i} \rangle \right)^2 - O(M^{-d})$$

$\Rightarrow$ Claim of Lemma □

The 2-Block estimate follows from

Lemma $D_{1i}(\sqrt{f}) \le C\ell^2$

Pf



$$D_{1i}(g) = \frac{1}{M^{2d}} \sum_{\substack{x\in B_1 \\ y\in B_i}} \int (\partial_{xy} g)^2 d\nu$$

Write

$$\partial_{xy} g = g(\eta^{xy}) - g(\eta)$$

$$= \sum_{i=1}^{A} \partial_{x_{i-1} x_i} g \qquad x_0 = x, \; x_A = y$$

where $A \le d\ell N$.

By Schwartz: $(\partial_{xy} g)^2 = (\sum_1^A \partial g)^2 \le A \sum_1^A (\partial g)^2$

Now take $g = \sqrt{f}$ = translation invariant.

Then
$$\int (\partial_{x_{i-1}x_i} \sqrt{f})^2 d\nu = \text{constant in } i$$

$$= \frac{1}{d N^d} \sum_{\substack{x,y\in\Lambda_N \\ |x-y|=1}} \int (\partial_{xy}\sqrt{f})^2 d\nu$$

$$= \frac{1}{N^d} D(\sqrt{f}) \le \frac{C}{N^d} \frac{N^{d-2}}{\ell} \le \frac{CN^{-2}}{\ell}$$

So
$$D_{1i}(\sqrt{f}) < A^2 \frac{CN^{-2}}{\ell} \le \frac{C}{\ell} \ell^2 \qquad \square$$