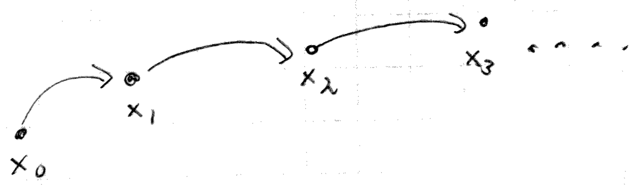
1. Markov Chains



$x_i \in S$ *states*, jump $x_i \to x_{i+1}$, i "time" $\in \mathbb{N}$

Probability of jump $p(x_i, x_{i+1})$ depends on x_i,

not on x_j, $j < i$.

Let S finite or countable set (will generalize later)

Def $(p(x,y))_{x,y \in S}$ is a *stochastic matrix* if

a) $p(x,y) \geq 0 \quad \forall x,y \in S$

b) $\sum_{y \in S} p(x,y) = 1$

Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Let X_n, $n = 0, 1, \ldots$ be random variables on $(\Omega, \mathcal{F}, P)$ with values in S; $X_n : \Omega \to S$ measurable where S is equipped with σ-algebra of all subsets of S.

Def X_n is a *Markov chain* w. *transition probability* $p(x,y)$ if for all $n \geq 0$, all $x_0, \ldots, x_{n-1}, x, y \in S$

$$P(X_{n+1} = y \mid X_n = x, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0) = p(x,y) \quad (1)$$

(where $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$ is conditional probability, and demand (1) for x_i s.t. $P(X_n = x, \ldots, X_0 = x_0) \neq 0$)

So future depends only on present.

Now, given a matrix $p(x,y)$ and $x_0 \in S$, construct a Markov chain X_n on some (Ω, Σ, P):

1° To get $X_0, \dots, X_n$ take

$$\Omega_n = S^{n+1} = \{(s_0, \dots, s_n) \mid s_i \in S)$$

$$\Sigma_n = \{\text{subsets of } S^{n+1}\}$$

$$P(\omega) = \begin{cases} p(s_0,s_1)\, p(s_1,s_2) \dots p(s_{n-1},s_n) & s_0 = x_0 \\ 0 & s_0 \neq x_0 \end{cases}$$

where $\omega = (s_0, s_1, \dots, s_n)$. Clearly $\sum_{\omega \in \Omega_n} P(\omega) = 1$

Now define $X_m(\omega) = s_m$, $0 \le m \le n$. Then

$$P(X_{n+1} = s_{n+1} \mid X_n = s_n \dots X_0 = s_0) = \frac{P(X_{n+1} = s_{n+1}, \dots, X_0 = s_0)}{P(X_n = s_n, \dots, X_0 = s_0)} = p(s_n, s_{n+1})$$

i.e. (1) holds

2° To get $(X_n)_{n<\infty}$ let

$$\Omega = S^{\mathbb{N}} = \{(s_0, s_1, \dots) \mid s_i \in S)$$

Def Cylinder sets in Ω are all sets of form

$$A^{n_0 \dots n_m}_{u_0 \dots u_m} \equiv \{\omega = (s_0, s_1, \dots) \mid s_{n_i} = u_i,\ i = 0, \dots, m\}$$

where $m \geq 0$, $(u_0, \dots, u_m) \in S^{m+1}$, $n_0 < n_1 < n_2 \dots < n_m$
i.e. fix finite # of coordinates. Let $\Sigma_0 = \{\text{cylinder sets}\}$.

$\Sigma \equiv$ σ-algebra generated by cylinder sets
$\equiv$ smallest σ-algebra containing them

Remark: It is enough to consider sets with $n_i = i$ (Why?), put $A_{u_0 \dots u_m} \equiv A^{012\dots m}_{u_0 \dots u_m}$

Define functions $X_n : \Omega \to S$ by

$$X_n(\omega) = s_n \quad (\omega = (s_0, s_1, \dots))$$

and so $A_{u_0 u_1 \dots u_m} = \{\omega \mid X_i(\omega) = u_i,\ i = 0, \dots m\}$

Let $P^x : \Sigma_0 \to S$ be

$$(2) \qquad P^x(A_{u_0 \dots u_m}) = \begin{cases} p(u_0,u_1)\,p(u_1,u_2) \cdots p(u_{m-1},u_m) & u_0 = x \\ 0 & u_0 \neq x \end{cases}$$

We would like to extend this to Σ as a prob. measure.

<u>Need:</u> <u>Kolmogorov extension thm</u>

Let
- S be complete separable metric space
- I countable set, $I = \{i_1, i_2, \dots\}$ (ex: $\mathbb{N}, \mathbb{Z}^d, \dots$)
- $\Omega = S^I = \{\omega : I \to S\}$
- Σ = smallest σ-algebra on Ω containing sets $\{\omega : \omega(i) \in B\}$ where $i \in I$, $B \subset S$ Borel set

Let, for each n be given prob. measure μ_n on S^n.
Assume (μ_n) are <u>consistent</u> i.e.

$$\mu_{n+1}(A \times S) = \mu_n(A) \qquad \forall A \subset S^n \text{ Borel set}$$

Then $\exists$ prob measure P on (Ω, Σ) s.t.

$$P(\omega \in \Omega : (\omega(i_1), \omega(i_2), \dots, \omega(i_n)) \in A)$$

$$= \mu_n(A) \qquad \forall A \subset S^n \text{ Borel}, \quad \forall n < \infty \qquad \square$$

Since $\sum_{u_{n+1} \in S} p(u_n, u_{n+1}) = 1$, (2) are consistent.

Thus, on (Ω, Σ, P^x) can define Random variable

$X_n(\omega) = s_n$ and X_n is a Markov chain

2. Non-interacting particles

<u>Random walk on $\mathbb{Z}^d$</u> is Markov chain w. $S = \mathbb{Z}^d$, and some $p(x,y)$ stochastic matrix.

<u>Simple random walk</u>: $p(x,y) = \begin{cases} 1/2d & |x-y|=1 \\ 0 & \text{otherwise} \end{cases}$

<u>Translation invariant</u> RW: $p(x,y) = p(0, y-x) \equiv \tilde{p}(x-y)$ $\forall x,y$. <u>Let us assume</u> also <u>finite range</u> $\tilde{p}(x) = 0$ $|x|>R$ some $R<\infty$. <u>Example</u> SRW.

<u>K non-interacting particles on torus</u>

Let $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$ = integers modulo N
i.e. $x \in \mathbb{Z}$ and $x \sim x + mN$ $\forall m \in \mathbb{Z}$

Random walk on $(\mathbb{T}_N)^d$: $S = (\mathbb{T}_N)^d$, take p as above, $R < N$.

Let P^x be the prob. measure, X_n the Markov chain

<u>Easy to see</u>: X_n have the same law as $x + Y_n$ when Y_n have the law P^0 i.e. Y_n is a walk starting at origin.

Let $Y^1, \dots, Y^K$ be K independent random walks starting at 0. Let, for $x_0^i \in \mathbb{T}_N^d$ $i=1,\dots,K$

$$X_n^i = x_0^i + Y_n^i \quad (\text{mod } N)$$

be K walks $X_0^i = x_0^i$.

Hence $\underset{\sim}{X}_n = (X_n^1, \dots, X_n^K) \in (\mathbb{T}_N^d)^K$ is a Markov process with transition probability

$$p(\underset{\sim}{x}, \underset{\sim}{x}') = \prod_{i=1}^{K} p(x_i, x_i') = \prod_{i=1}^{K} \tilde{p}(x_i' - x_i)$$

Think of $(X_n^i)_{i=1}^K$ as positions of K particles moving in $\mathbb{T}_N^d$.

Let $\eta_n(x)$ = # of particles at x on time n $= \sum_{i=1}^{K} \mathbb{1}(X_n^i = x)$

Prop η_n defines a Markov process with state space $S \equiv \mathbb{N}^{\mathbb{T}_N^d}$

Proof. Since η_n is a function of $\underset{\sim}{X}_n$ we have

$$(*) \quad P(\eta_{n+1} = \eta' \mid \underset{\sim}{X}_m = \underset{\sim}{x}_m, m \le n) = P(\eta_{n+1} = \eta' \mid \underset{\sim}{X}_n = \underset{\sim}{x}_n)$$

when $\eta' \in \mathbb{N}^{\mathbb{T}_n^d}$. This is $\neq 0$ only if $\sum_{x \in \mathbb{T}_N^d} \eta'(x) = K$.

Given $\underset{\sim}{x}' \in (\mathbb{T}_N^d)^K$ denote $\eta^{\underset{\sim}{x}'} \in S$ the function $\eta^{\underset{\sim}{x}'}(x) = \sum_{i=1}^{K} \mathbb{1}(x_i' = x)$. So

$$(*) = \sum_{\underset{\sim}{x}' \in (\mathbb{T}_N^d)^K} \mathbb{1}(\eta^{\underset{\sim}{x}'} = \eta') \prod_{i=1}^{K} \tilde{p}(x_i' - x_i)$$

Now $$\prod_{i=1}^{K} \tilde{p}(x_i' - x_i) = \prod_{x:\, \eta^{\underset{\sim}{x}}(x) \neq 0} \; \prod_{i:\, x_i = x} \tilde{p}(x_i' - x) \quad (**)$$

Let $\underset{\sim}{y}$ be s.t. $\eta^{\underset{\sim}{y}} = \eta^{\underset{\sim}{x}}$. Then $\{x : \eta^{\underset{\sim}{x}}(x) \neq 0\}$ $= \{x : \eta^{\underset{\sim}{y}}(x) \neq 0\}$ and $\{i : x_i = x\} = \{i : y_i = x\}$ (particles at x are just relabeled). Hence $(**)$ is the same for $\underset{\sim}{y} \Rightarrow (*)$ depends only on $\eta = \eta^{\underset{\sim}{x}}$ $\square$

Invariant measures

Let $X(t)$ be Markov chain. The probability distribution of $X(t)$ equals $P^x(X(t)=y) = p^t(x,y)$, p^t matrix product

Suppose $X(0)$ is also distributed w. some prob measure μ; $P(X(0)=x) = \mu(x)$, $\sum_{x\in S} \mu(x) = 1$. Then set

$$\mu_t(y) \equiv P_\mu(X(t)=y) = \sum_{x\in S} \mu(x)\, P^x(X(t)=y) = \sum_x \mu(x) p^t(x,y)$$

We say μ is invariant if $\mu_t = \mu$ i.e. if

$$\mu P = \mu \quad \text{(matrix product)}$$

Example RW on $\mathbb{T}_N^d$: $\mu(x) = \frac{1}{N^d}$ is invariant

RW on $\mathbb{Z}^d$: no inv. prob. measures

How about η? Let $X^1, \dots, X^k$ be independent, uniformly distributed on $\mathbb{T}_N^d$. How is $\eta(x) = \sum_{i=1}^{k} \mathbb{1}(X^i = x)$ distributed? We'll proceed indirectly.

Recall: Poisson distribution with mean α is prob measure p_α on $\mathbb{N}$:

$$p_\alpha(k) = e^{-\alpha} \frac{\alpha^k}{k!}$$

Prop Let μ_α be the prob. measure $\prod_{x\in \mathbb{T}_N^d} p_\alpha(\eta(x))$ on $\mathbb{N}^{\mathbb{T}_N^d}$, i.e. $\eta(x)$ are i.i.d. Poisson α. Then μ_α is invariant measure for the η process.

Proof Let P^{η^0} be the probability measure for the η - process i.e. P^{η^0} is measure on

$$(\mathbb{N}^{\mathbb{T}_N^d})^{\mathbb{N}} = \{\eta : \mathbb{N} \to \mathbb{N}^{\mathbb{T}_N^d} ; \eta_n = \{\eta_n(x)\}_{x\in\mathbb{T}_N^d}\}$$

and η_0 is the fixed configuration η^0.

Concretely, $P^{\eta^0}(\eta_{n_1} = \eta^{(1)}, \eta_{n_2} = \eta^{(2)}, \dots \eta_{n_k} = \eta^{(k)})$ is obtained by taking $K = \sum_{x\in\mathbb{T}_N^d} \eta^0(x)$ independent random walkers $X_n^{y_i,i}$ $i = 1,\dots,K$, starting at $y_i = X_0^{y_i,i}$ where y_i are the points where $\eta^0(x) \neq 0$ and for each such x, $y_i = x$ for $\eta^0(x)$ number of i.

Recall If k_1, k_2 are independent Poisson random variables, means α_1, α_2 then $k_1 + k_2$ is Poisson$(\alpha_1+\alpha_2)$ [Pf: $P(k_1+k_2=k) = \sum_{k_1=0}^{k} \frac{\alpha_1^{k_1}\alpha_2^{k-k_1}}{k_1!(k-k_1)!} e^{-\alpha_1-\alpha_2}$

$$= \frac{e^{-\alpha_1-\alpha_2}}{k!} \sum_{k_1} \binom{k}{k_1} \alpha_1^{k_1}\alpha_2^{k-k_1} = \frac{e^{-\alpha_1-\alpha_2}}{k!}(\alpha_1+\alpha_2)^k \ ]$$

Thus, the # of particles $k = \sum \eta^0(x)$ is Poisson$(N^d\alpha)$

Let $P^{(\alpha)}$ be the measure on $(\mathbb{N}^{\mathbb{T}_N^d})^{\mathbb{N}}$ where η_0 is distributed w. μ_{α}.

We compute now the distribution of η_n via its Laplace transform:

$$\mathbb{E}^{(\alpha)} \exp\Big[-\sum_{x\in \mathbb{T}_N^d} \lambda(x)\, \eta_n(x)\Big] \qquad (A)$$

Here $\eta_n(x) = \sum_{y\in \mathbb{T}_N^d} \sum_{i=1}^{\eta_0(y)} \mathbb{1}(X_n^{y,i}=x)$ so

$$\sum_x \lambda(x)\,\eta_n(x) = \sum_y \sum_{i=1}^{\eta_0(y)} \lambda(X_n^{y,i})$$

so

$$(A) = E^{(\alpha)} \prod_{y\in \mathbb{T}_N^d} \prod_{i=1}^{\eta_0(y)} e^{-\lambda(X_n^{y,i})}$$

Now, $\eta_0(y)$ are independent and X_n are also, so

$$= \prod_y E_{\mu_\alpha}\Big(E_{RW}\, e^{-\lambda(X_n^{y,1})}\Big)^{\eta_0(y)} \qquad (B)$$

where E_{RW} is expectation in RW measure and

E_{μ_α} for the poisson(α) variable $\eta_0(y)$

Now, let k be Poisson(α), then

$$\mathbb{E}\, e^{-ak} = \sum_{k=0}^{\infty} e^{-\alpha}\frac{\alpha^k}{k!} e^{-ak} = e^{\alpha(e^{-a}-1)}$$

so

$$(B) = \prod_y \exp \alpha\Big(E\, e^{-\lambda(X_n^{y,1})} - 1\Big)$$

$$= \prod_y \exp \alpha\Big(E\, e^{-\lambda(y+X_n^{0,1})} - 1\Big)$$

But

$$E\, e^{-\lambda(y+X_n^{0,1})} = \sum_{x\in \mathbb{T}_N^d} e^{-\lambda(y+x)}\, p^n(0,x)$$

so

$$\sum_y E\, e^{-\lambda(y+X_n^{0,1})} = \sum_y \sum_x e^{-\lambda(y+x)}\, p^n(0,x)$$

$$\underset{y+x=\tilde{y}}{=} \sum_{\tilde{y}} e^{-\lambda(\tilde{y})} \underbrace{\sum_x p^n(0,x)}_{=1} = \sum_y e^{-\lambda(y)}$$

So finally

$$E^{(\alpha)} \exp\Big[-\sum_x l(x)\,\eta_n(x)\Big] = \exp \sum_x \alpha\,(e^{-l(x)}-1)$$

Thus $\eta_n(x)$ are i.i.d. poisson(α) $\square$

<u>Remarks</u> 1. α is the <u>density</u> of particles

$$\mathbb{E}\,\eta(x) = \alpha$$

2. By weak law of large numbers

$$\lim_{N\to\infty} \frac{1}{N^d} \sum_{x\in\mathbb{T}_N^d} \eta(x) = \alpha$$

in probability (i.e. $\forall \varepsilon > 0$, $P\big(\big|\frac{1}{N^d}\sum \eta(x) - \alpha\big| > \varepsilon\big)$ $\to 0$ as $N \to \infty$)

3. The measures μ_α are not ergodic (see later def.): the number of particles $\sum_x \eta(x)$ is conserved.

Ergodic measure is

$$\nu_K = \mu_\alpha\big(\cdot \mid \textstyle\sum \eta(x) = K\big)$$

(This is α-independent, why?)

which is the one induced to η from the uniform measures for the K particles (prove!).

<u>Local Equilibrium</u>

Suppose start the system at a state μ which is not stationary. What happens to μ_n = Prob distribution of X_n as $n \to \infty$? Does it tend to some μ_α?

If so, how? Study this in a continuum limit where lattice spacing $\to 0$ and μ is spatially slowly varying. Lattice = Micro, study Macro behaviour.

Let $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d \cong [0,1)^d$ periodic box.

Embed $\mathbb{T}_N^d \subset \mathbb{T}^d$ as $x \in \mathbb{T}_N^d \to \frac{x}{N} \in \mathbb{T}^d$, lattice spacing $= 1/N$.

Fix $\rho : \mathbb{T}^d \to \mathbb{R}$ smooth function.

Let ν_ρ^N be the measure on $\mathbb{N}^{\mathbb{T}_N^d}$ where $\eta(x)$, $x \in \mathbb{T}_N^d$ are independent, Poisson with parameter $\rho(x/N)$. Thus α is not constant but varies w. x:

$$\mathbb{E}_{\nu_\rho^N} \eta(x) = \rho(x/N)$$

We try to take $N \to \infty$ and study time evolution.

ν_ρ^N is a measure on $\mathbb{N}^{\mathbb{T}_N^d}$.

What does ν_ρ^N look like close to $u \in \mathbb{T}^d$?

I.e. close to $x = [Nu] \in \mathbb{T}_N^d$? Test it there:

$$\mathbb{E}_{\nu_\rho^N} \exp\Big[-\sum_{|y|<R} f(y)\, \eta([Nu]+y)\Big]$$

$$= \exp \sum_y \rho\Big(\frac{[Nu]+y}{N}\Big)\big(e^{-f(y)}-1\big) \xrightarrow[N\to\infty]{} \exp \sum_{|y|<R} \rho(u)\big(e^{-f(y)}-1\big)$$

$$= \mathbb{E}_{\nu_{\rho(u)}} e^{-\sum_{|y|<R} \lambda(y)\eta(y)} \qquad (*)$$

where $\nu_{\rho(u)}$ is i.i.d Poisson$(\rho(u))$ measure on $\mathbb{N}^{\mathbb{Z}^d}$.

<u>Aside on measure theory</u>

We need to talk about measures in spaces like $\mathbb{N}^{\mathbb{Z}^d}$ or $\{0,1\}^{\mathbb{Z}^d}$. When $\mathbb{Z}^d$ is replaced by a finite set, these sets are countable (or finite) and measures are defined on σ-algebra of all sets.

In general, $\mathbb{N}^{\mathbb{Z}^d}$ and $\{0,1\}^{\mathbb{Z}^d}$ are complete separable metric spaces i.e. <u>Polish spaces</u>.

Let I countable, Y metric space. Give Y^I the product topology (i.e. where projections to coordinates are continuous). Y^I is metrizable:

Write $I = \{i_1, i_2, \dots\}$, $\underset{\sim}{y} = (y_{i_1}, y_{i_2}, \dots) \in Y^I$,

$$d(\underset{\sim}{y}, \underset{\sim}{y}') = \sum_j 2^{-j} \hat{d}(y_{i_j}, y'_{i_j})$$

If Y is Polish then Y^I is. If Y is compact Y^I

Borel σ-algebra $\mathcal{B}(Y)$ is generated by open sets.

$\mathcal{M}_1(Y) \equiv$ Prob. measures on Y.

$C_b(Y) =$ Bounded continuous functions on Y.

where r metric of Y, $y = (y_i)_{i \in \mathbb{Z}^d}$.

<u>Show</u>: This metric gives same topology (i.e. product topo) and is complete, separable. If Y is <u>compact</u>, so is Y^I.

<u>So Y^I is Polish too.</u>

Let Y be Polish. Set $\mathcal{B}(Y) \equiv$ Borel σ-algebra.

Let $\mathcal{M}_1(Y) =$ set of probability measures on $(Y, \mathcal{B}(Y))$

We can make $\mathcal{M}_1(Y)$ a metric space:

Given two measures $\mu_1, \mu_2 \in \mathcal{M}_1(Y)$

define the <u>Prohorov metric</u>

$$\rho(\mu_1, \mu_2) = \inf\{\varepsilon > 0 \mid \mu_1(A) \le \mu_2(A^\varepsilon) + \varepsilon \text{ for all closed sets } A \in \mathcal{B}(Y)\}$$

where $A^\varepsilon = \{y \in Y \mid \inf_{x \in A} d(x,y) < \varepsilon\}$

[Note this is symmetric i.e. $\rho(\mu_1, \mu_2) = \rho(\mu_2, \mu_1)$:

suppose $\mu_1(A) \le \mu_2(A^\varepsilon) + \varepsilon$ $\forall A$ closed. Put

$B = Y \setminus A^\varepsilon$ so B is closed and

$A \subset Y \setminus B \subset Y \setminus B^\varepsilon$ so

$$\mu_1(A^\varepsilon) = 1 - \mu_1(B) \geq 1 - \mu_2(B^\varepsilon) - \varepsilon = \mu_2(Y \setminus B^\varepsilon) - \varepsilon$$

$$\geq \mu_2(A) - \varepsilon \quad \text{so} \quad \mu_2(A) \le \mu_1(A^\varepsilon) + \varepsilon \quad \square]$$

<u>Excercise</u> prove triangle inequality!

The main properties of this metric are:

1. If (Y, d) is Polish space then $(M_1(Y), \varrho)$ is also

2. If Y is compact so is $M_1(Y)$.

A more concrete characterization of this metric is via Weak convergence:

Def $\mu_n \to \mu$ weakly $(\mu_n, \mu \in M_1(Y))$

if for all $f \in C_b(Y)$ = Bounded continuous fns

$$\int f \, d\mu_n \to \int f \, d\mu \qquad (A)$$

One has: If Y is Polish then $(A) \Leftrightarrow$ convergence $\varrho(\mu_n, \mu) \to 0$.

Thus weak convergence of measures = convergence in ϱ-metric.

Now if you want to prove $\mu_n \to \mu$ one way is to show $\{\mu_n\}$ is relatively compact so $\exists\, \mu_{n_j}$ that converges $\mu_{n_j} \xrightarrow[j\to\infty]{} \mu$.

At most one limit point μ.

So when is $A \subset M_1(Y)$ relatively compact?

Answer $\Leftrightarrow A$ is tight i.e. $\forall \varepsilon\, \exists K \subset Y$ compact s.t. $\inf_{\mu \in A} \mu(K) \geq 1-\varepsilon$ i.e. A

lives to arbitrary accuracy on a compact set.

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Go back to $(*)$ on p. 11.

We can embed

$$\mathbb{N}^{\mathbb{T}_N^d} \subset \mathbb{N}^{\mathbb{Z}^d}$$

by extending periodically the configuration in $\mathbb{T}_N^d$ to $\mathbb{Z}^d$.

Thus we may view $\nu_g^N \in \mathcal{M}_1(\mathbb{N}^{\mathbb{Z}^d})$.

Let τ_y, $y \in \mathbb{Z}^d$ be translation

$$\tau_y : \mathbb{N}^{\mathbb{Z}^d} \to \mathbb{N}^{\mathbb{Z}^d} : (\tau_y \eta)(x) = \eta(x+y)$$

and for $\mu \in \mathcal{M}_1(\mathbb{N}^{\mathbb{Z}^d})$ put

$$\tau_y \mu : \quad \int f(\eta)\,(\tau_y \mu)(d\eta) \equiv \int f(\tau_y \eta)\,\mu(d\eta)$$

Then $(*)$ gives:

$$\tau_{[Nu]}\, \nu^N_\rho \xrightarrow[\text{weakly}]{} \nu_{\rho(u)}$$

i.e. looking in a (micro) nhood of $u \in \mathbb{T}^d$

we see Poisson measure of average $\rho(u)$.

<u>Hydrodynamic limit</u>

Let us study now the distribution of $\eta(t)$.

Compute the Laplace transform as before (p. 8)

$$E_{\nu^N_\rho} \exp\Big[-\sum_{x \in \mathbb{T}^d_N} \lambda(x)\, \eta_t(x)\Big]$$

$$= E_{\nu^N_\rho} \exp\Big[-\sum_y \sum_{i=1}^{\eta_0(y)} \lambda(X_n^{y,i})\Big]$$

$$= \prod_y \mathbb{E}_{\text{POISSON}(\rho(\frac{y}{N}))} \left(\mathbb{E}_{RW}\, e^{-\lambda(X_n^{y,1})}\right)^{\eta_0(y)}$$

$$= \exp \sum_y \rho\left(\tfrac{y}{N}\right)\left(\mathbb{E}_{RW}\, e^{-\lambda(X_n^{y,1})} - 1\right)$$

$$= \exp \sum_y \rho\left(\tfrac{y}{N}\right)\left(\sum_x p^n(0,x)\left(e^{-\lambda(y+x)} - 1\right)\right)$$

$$\underset{x \to x-y}{=} \exp \sum_x \left(e^{-\lambda(x)} - 1\right) \sum_y p^n(0, x-y)\, \rho\left(\tfrac{y}{N}\right)$$

Ie. got Poisson measure with parameters

$$\sum_y p^n(0, x-y)\, \rho(y/N) \equiv \psi_n^N(x)$$

The macroscopic profile at time n is, $(x \in \mathbb{T}_1^d)$

$$\rho_n^N(x) \equiv \psi_n^N(Nx) = \sum_y p^n(0, Nx-y)\, \rho(y/N)$$

$$= \sum_y p^n(0, y)\, \rho(x - y/N)$$

Let $p(x,y) = 0$, $|x-y| > R$. Then $p^n(0,y) = 0$ if $|y| > nR$. So

$$\rho_n^N(x) \xrightarrow[N\to\infty]{} \rho(x) \sum_y p^n(0,y) = \rho(x)$$

and profile does not change.

<u>Scale time</u>: put $n = [Nt]$, $t \in \mathbb{R}$. Now

$p^n(0,y) = \text{Prob}(X_n = y)$, and

$$X_n = \sum_{m=1}^n X_m - X_{m-1}$$

where $Z_m \equiv X_m - X_{m-1}$ are i.i.d. $P(Z_m = z) = p(0,z)$

By Law of Large numbers

$$\lim_{n\to\infty} P\left(\left|\frac{X_n}{n} - m\right| \leq \varepsilon\right) = 1 \qquad \forall \varepsilon > 0$$

$$m = \mathbb{E}(Z_n) = \sum_z z\, p(0,z)$$

so
$$\lim_{N\to\infty} P\left(\left|\frac{X_{[Nt]}}{N} - mt\right| \le \varepsilon\right) = 1$$

i.e. $\lim_{N\to\infty} \sum_{y:\, |y/N - mt| \geq \varepsilon} p^{[Nt]}(0,y) = 0 \implies$

$$\rho^N_{[Nt]}(x) = \psi_{[Nt]}(Nx) \xrightarrow[N\to\infty]{} \rho(x - mt)$$

i.e. the $t=0$ profile moves with speed m. We got ρ

<u>Prop</u>

$\lim_{N\to\infty} \rho^N_{[Nt]}(x) = \rho_t(x)$ and satisfies the eqn

$$\partial_t \rho + m\cdot\nabla\rho = 0$$

where $m = \sum_x p(0,x)\,x$.

Now, suppose $m=0$. Then can look at different scaling: take $n = [N^2 t]$. Then

$$\rho^N_{[N^2t]}(x) = \sum_{y\in\mathbb{T}^N_d} p^{[N^2t]}(0, Nx - y)\,\rho(y/N)$$

Now, by central limit theorem

$$p^{[N^2t]}(0, [Nx]) \xrightarrow[N\to\infty]{} p_t(x)$$

pointwise $x\in\mathbb{T}_d$ where $p_t(x)$ is Gaussian distribution with covariance $t\sigma$,

$$\sigma_{ij} = \sum_{x \in \mathbb{Z}^d} p(0,x)\, x_i x_j .$$

This means

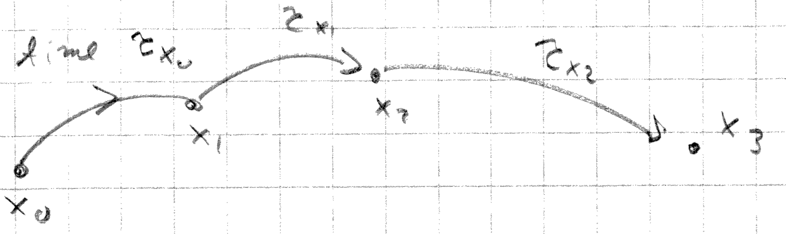
$$\partial_t\, \rho_t(x) = \sum_{ij} \sigma_{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}\, \rho_t(x)$$

so $\lim_{N\to\infty} \rho^N_{[N^2 t]}(x)$ exists $\equiv \rho_t(x)$ and

$$\begin{cases} \partial_t\, \rho_t(x) = \sum \sigma_{ij} \dfrac{\partial^2}{\partial x_i \partial x_j}\, \rho_t(x) \\ \rho_0(x) = \rho(x) \end{cases}$$

Continuous time Markov Chains

Let state space S be countable.



Wait random time τ_{x_n} at x_n and then jump to x_{n+1}

1. τ_x should depend only on x, not on the time already spent at x.

Recall: Let T be exponential r.v. i.e

$$P(T > t) = e^{-ct}, \quad c^{-1} \text{ is mean } \mathbb{E}T = c^{-1}$$

Then $$P(T > t+s \mid T > s) = \frac{P(T > t+s)}{P(T>s)} = e^{-ct}$$

so take τ_x expo, mean $1/c(x)$

$$P(\tau_x > t) = e^{-c(x)t}$$

2. When the clock rings, jump to y with proba $p(x,y)$, stochastic matrix

Construction:

1. Markov chain $Y_n \in S$, trans. prob $p(x,y)$ on proba space Ω_1

2. Function $c : S \to \mathbb{R}_+$

3. I. i. d. random variables $(\tau_j)_{j=0}^{\infty}$

$$P(\tau_j > t) = e^{-t}$$

on prob. space $(\Omega_2, \Sigma_2, P_2)$

Full space $(\Omega_1 \times \Omega_2, \Sigma_2 \times \Sigma_2, P_1 \times P_2) \equiv (\Omega, \Sigma, P)$

$$x = Y_0 \xrightarrow[\frac{\tau_0}{c(Y_0)}]{} Y_1 \xrightarrow[\frac{\tau_1}{c(Y_1)}]{} Y_2 \to \ldots$$

$$T_n := \sigma_0 + \sigma_1 + \ldots + \sigma_{n-1} \quad , \quad \sigma_i = \tau_i / c(Y_i)$$

$$X_t = Y_n \qquad T_n \le t < T_{n+1} \quad n = 0, 1, \ldots$$

defines a stochastic process. P^x the law $(x = Y_0)$

Let $\quad p_t(x, y) = P^x(X_t = y)$

Then one can show

$$P^x(X_{t_0} = x_0, X_{t_1} = x_1, \ldots, X_{t_n} = x_n) = p_{t_0}(x, x_0)\, p_{t_1 - t_0}(x_0, x_1) \cdots p_{t_n - t_{n-1}}(x_{n-1}, x_n)$$

which implies Markov property

$$P^x(X_{t_n} = x_n \mid X_{t_{n-1}} = x_{n-1} \ldots) = p_{t_n - t_{n-1}}(x_{n-1}, x_n)$$

<u>Path space</u>

We want to discuss convergence of such processes again. Now X_t defines $X : [0, \infty) \to S$, so our path measures are on function spaces.

We put suitable metric on such paths.

Main headache: paths are discontinuous.

We have defined them to be continuous from the right with left limits

Def $X \in D_S[0,\infty)$ if $\forall t \geq 0$ $\lim_{s \downarrow t} X(s) = X(t)$

and $\lim_{s \uparrow t} X(s) \equiv X(s-)$ exists

Lemma $X \in D_S[0,\infty)$ has countable # of discontinuities

Pf Let $D_n = \{ t > 0 \mid d(X(t), X(t-)) > \frac{1}{n} \}$

D_n has no limit points: if $t_i \in D_n$, $t_i \to t \in D_n$ then we would have $\tilde{t}_i \uparrow t$ or $\tilde{t}_i \downarrow t$ s.t. $d(X(\tilde{t}_{2i}), X(\tilde{t}_{2i+1})) > 1/n$

But $\lim_{s \uparrow t} X(s)$ and $\lim_{s \downarrow t} X(s)$ exist,

Thus D_n is countable and $\bigcup_{n=0}^{\infty} D_n$ is too $=$ set of discontinuities $\square$

Put a metric on $D_S[0,\infty)$ s.t. it becomes Polish if S is. Idea: D_S is not that different from $S^{\mathbb{N}}$ except the times of jumps are varying. Say to paths $X(t)$, $X'(t)$ are close if their jump times are close and jumps nearly same. So want to compare x and x' at nearby times. Let Λ be the set of uniformly Lipschitz homeomorphisms of $[0,\infty)$ i.e. $\lambda \in \Lambda$ is $\lambda : [0,\infty) \to [0,\infty)$ is continuous, strictly increasing, $\lambda(0)=0$, $\lambda(\infty)=\infty$ and

$$e^{-\gamma} < \left| \frac{\lambda(t)-\lambda(s)}{t-s} \right| < e^{\gamma}$$

for some $\gamma \geq 0$, $\forall t > s \geq 0$. [Note: if $\gamma = 0$ $\lambda(s)=s$]. For $x, y \in D_S[0,\infty)$, $\lambda \in \Lambda$ and $s > 0$ put

$$\rho_s(x,y,\lambda) = \sup_{0 \leq t \leq s} d(x(t), y(\lambda(t)))$$

and then define

$$\rho(x,y) = \inf_{\lambda \in \Lambda} \left\{ \gamma(\lambda) + \int_0^\infty e^{-s} \rho_s(x,y,\lambda)\,ds \right\}$$

<u>Exercise</u> a) ρ is a metric

b) Let $x, x_n \in D_S[0,\infty)$. Then

$$\lim_{n\to\infty} \rho(x_n, x) = 0 \qquad \text{if and only if}$$

there exist λ_n with $\gamma(\lambda_n) \to 0$ (i.e. $\lambda_n \to id$)

and $\lim\limits_{n\to\infty} \rho_s(x_n, x, \lambda_n) = 0$ $\forall s$ where $x(s)$ is continuous. In particular

$$\lim_{n\to\infty} x_n(u) = \lim_{n\to\infty} x_n(u-) = x(u)$$

for all u where x is continuous.

ρ is called the Skorohod metric and (D_S, ρ) the Skorohod space.

The main result is

Prop If (S,d) is Polish, $(D_S[0,\infty), \rho)$ is ~~too~~.

Proof is not hard. A countable dense set is step functions taking values in a countable dense set in S and jumps at rational times. □

Let $\mathcal{B}(D_S)$ be the Borel σ-algebra on $D_S[0,\infty)$. Let $\mathcal{C}(D_S)$ be the σ-algebra generated by cylinder sets i.e. the one generated by $\pi_t^{-1}(B)$, B Borel in S,

$$\pi_t : D_S \to S \quad , \quad \pi_t(x) = x(t)$$

Prop If S is Polish then $\mathcal{B}(D_S) = \mathcal{C}(D_S)$. □

We have got the following.

$X_t(\omega)$ with $\omega \in \Omega$ = Prob space of (Y_n, τ_n)

takes values in $D_S([0,\infty))$.

Hence: $X_\bullet$ is a D_S-valued random variable i.e. $X_\bullet : (\Omega, \Sigma) \to (D_S, \mathcal{B}(D_S))$ is measurable (check that it is!).

Let $\mathbb{P}^x$ be its distribution i.e. $\mathbb{P}^x$ is the probability measure on $(D_S, \mathcal{B}(D_S))$ defined by

$$\mathcal{P}^x(A) = P^x(X_\bullet \in A)$$

$$\mathcal{E}^x \equiv \text{expectation in } \mathcal{P}^x.$$

Lets denote the elements of D_S by $\xi(t)$.

Then transition probability

$$p_t(x,y) = \mathcal{P}^x(\xi(t) = y)$$

Define <u>shift</u> θ_t on D_S:

$$(\theta_t \xi)(s) = \xi(t+s)$$

Let

$$\mathcal{F}_t = \sigma\text{-algebra generated by } \xi(s),\ s \le t$$

We can then reformulate <u>Markov property</u>:

Let $f : D_S \to \mathbb{R}$ measurable. Then

$$\mathcal{E}^x(f \circ \theta_t \mid \mathcal{F}_t) = \mathcal{E}^{\xi(t)}(f) \qquad (M)$$

Explanation

$\mathcal{E}^x(f \mid \mathcal{F}_t)$ is a $\mathcal{F}_t$ measurable r.v.

i.e. a measurable function of the path

$\{\xi(s) \mid s \in [0,t]\}$. The LHS is such function.

The RHS is $\mathcal{E}^y(f)$ evaluated at the

random point $y = \xi(t)$.

Proof Enough to consider $f = \mathbb{1}_A$ $A \in \mathcal{F}$

(we denote $\mathcal{B}(D_S)$ by $\mathcal{F}$), i.e.

$$\mathcal{E}^x(\mathbb{1}_A \circ \theta_t \mid \mathcal{F}_t) \stackrel{?}{=} \mathcal{E}^{\xi(t)}(\mathbb{1}_A)$$

This follows if show for all $B \in \mathcal{F}_t$

$$(*) \quad \mathcal{E}^x(\mathbb{1}_A \circ \theta_t \, \mathbb{1}_B) = \mathcal{E}^x(\mathcal{E}^{\xi(t)}(\mathbb{1}_A)\, \mathbb{1}_B)$$

[Homework: Why?]. This is easy for

cylinder sets A, B: Let $A = \{\xi \mid$

$\xi(t_i) \in A_i \;\; i = 1, \dots, n\}$, $B = \{\xi \mid \xi(s_i) \in B_i, \; i = 1 \dots m\}$

when $s_i \le t$. $LHS = P^x(\xi(t_i + t) \in A_i,$

$\xi(s_j) \in B_j) =$

$$= \sum_{x_i \in B_i} \sum_{y_i \in A_i} p_{s_1}(x, x_1)\, p_{s_2 - s_1}(x_1, x_2) \dots p_{s_m - s_{m-1}}(x_{m-1}, x_m)$$

$$p_{t_1 + t - s_m}(x_m, y_1)\, p_t(y_1, y_2) \dots p_{t_n - t_{n-1}}(y_{n-1}, y_n)$$

$$RHS = \sum_{y_i} \mathcal{E}^x \left(\mathbb{1}_B \; p_{t_1}(\xi(t), y_1)\, p_{t_2-t_1}(y_1, y_2) \cdots \right)$$

$$= \sum p_{s_1}(x, x_1) \cdots p_{s_m - s_{m-1}}(x_{m-1}, x_m)\, \underbrace{P_{t-s_m}(x_m, z)\, P_{t_1}(z, y_1)}_{\sum_z \,=\, P_{t+t_1-s_m}(x_m, y_1)}$$

$$P_{t_2-t_1}(y_1, y_2) \cdots$$

$= LHS$ <u>So OK for cylinder.</u>

Now let $\mathcal{L}$ be the set of A (resp B)

for which $(*)$ holds. We have

a) $D_S \in \mathcal{L}$
b) If $A_1, A_2 \in \mathcal{L}$ with $A_1 \subset A_2$ then $A_2 \setminus A_1 \in \mathcal{L}$
c) If $A_n \in \mathcal{L}$ and $A_n \uparrow A$ then $A \in \mathcal{L}$
(by monotone convergence thm).

Also Σ = cylindersets $\subset \mathcal{L}$ and Σ is

closed under intersections.

Then Dynkin π-λ thm $\Rightarrow$ σ-algebra generated by

Σ also $\subset \mathcal{L}$ $\square$

Generator of Markov process

Look at $E^x(f(X_t))$ as $t \to 0$.

$$E^x(f(X_t)) = \underbrace{E^x(\mathbb{1}_{T_1 > t} f(X_t))}_{(\mathrm{I})} + \underbrace{E^x(\mathbb{1}_{T_1 \leq t} f(X_t))}_{(\mathrm{II})}$$

(where T_1 is the time of the 2nd jump)

(I): either $T_0 > t$ and so $X_t = x$ or $T_0 \leq t$

$$= f(x) e^{-c(x)t} + (1 - e^{-c(x)t}) \sum_y p(x,y) f(y)$$

$$(\mathrm{II}) \leq \sup_x |f(x)| \, \underbrace{E^x(\mathbb{1}_{T_0 < t} \mathbb{1}_{T_1 < t})}_{(*)}$$

$$(*) = E^x(\mathbb{1}_{T_0 < t} \underbrace{E^x(\mathbb{1}_{T_1 < t} \mid T_0))}_{= \sum_y p(x,y)(1 - e^{-c(y)t})}$$

$$\leq \sum_y p(x,y)(1 - e^{-c(y)t})(1 - e^{-c(x)t})$$

<u>Assume</u> $\sup_x |c(x)| < \infty$. Get:

$$E^x(f(X_t)) = f(x) + (Lf)(x)\, t + o(t)$$

$$(Lf)(x) = c(x) \sum_y p(x,y)(f(y) - f(x))$$

L is the <u>generator</u> of our process. In particular

$$\limsup_{t \to 0} \sup_x \left| \frac{E^x(f(X_t)) - f(x)}{t} - Lf(x) \right| = 0$$

General Markov process

Above state space was countable. We need $S = \{0,1\}^{\mathbb{Z}^d}$ or $\mathbb{N}^{\mathbb{Z}^d}$ so we suppose state space Y is a metric space (and usually complete & separable i.e. Polish)

$D_Y = \{\omega : [0,\infty) \to Y \quad \text{RCLL (or CADLAG)}\}$

$\mathcal{F}$ = σ-algebra generated by $\pi_t(\omega) = \omega(t)$'s
(= Borel w.r.t. Skorokhod metric if Y POLISH)

$X_t(\omega) := \omega(t)$, $\mathcal{F}_t = \sigma(X_s, s \leq t)$

$\theta_t : D_Y \to D_Y \quad (\theta_t \omega)(s) = \omega(s+t)$

<u>Def</u> A Markov process is a family $\{P^x\}_{x \in Y}$ of prob measures $P^x \in \mathcal{M}_1(D_Y)$ with

a) $P^x(\{\omega \mid \omega(0) = x\}) = 1$

b) $\forall A \in \mathcal{F} \quad x \to P^x(A)$ is measurable in Y

c) $E^x(f \circ \theta_t \mid \mathcal{F}_t) = E^{\omega(t)}(f) \quad P^x$-a.e. ω

all $x \in Y$, f $\mathcal{F}$-measurable.

<u>Transition probability</u>

$$p(t, x, B) = P^x(X_t \in B)$$

is measurable in x and $B \to p(t, x, B)$ is in $\mathcal{M}_1(Y)$.

Markov property gives

$$p(t+s, x, B) = E^x(1_B(X_{t+s}))$$
$$= E^x\left[E^x(1_B(X_{t+s}) \mid \mathcal{F}_s)\right]$$
$$= E^x(E^{X(s)}(1_B(X_t)))$$
$$= E^x\left[p_t(X(s), B)\right]$$
$$= \int_Y p_s(x, dy)\, p_t(y, B)$$

= Chapman - Kolmogorov eqn.

<u>Semigroup</u> Let f $\mathcal{F}$-measurable, define

$$(S(t)f)(x) = E^x(f(X_t)) = \int_Y f(y)\, p(x, dy)$$

Then

(a) $\|S(t)f\|_\infty \leq \|f\|_\infty$

(b) $S(t) : L^\infty(Y) \to L^\infty(Y)$, $\|S(t)\| \leq 1$ linear operator

(c) Markov $\Rightarrow$ $S(t+s) = S(t)S(s) = S(s)S(t)$

(d) $S(0) = Id$

$S(t)$ <u>contraction semigroup</u> on $L^\infty(Y)$

The Markov process $\{P^x\}$ is uniquely determined by the transition probabilities:

P^x is determined by the

$P^x(X(t_i) \in A_i)$ $t_1 < t_2 < t_n$, $A_i \in \mathcal{B}(Y)$

$$= \int_{A_1 \times A_2 \times A_n} p_{t_1}(x, dy_1)\, p_{t_2 - t_1}(y_1, dy_2) \quad p_{t_n - t_{n-1}}(y_{n-1}, dy_n)$$

by π-λ theorem,

Given a semigroup $S(t): L^\infty \to L^\infty$ $\|S(t)\| \leq 1$,

$S(t) f \geq 0$ if $f \geq 0$ and $S(t) 1 = 1$ we

have

$$p_t(x, A) := (S(t) 1_A)(x)$$

is a prob measure. So such a semigp. determines

the process.

<u>Feller Processes</u> $\{P^x\}$ is <u>Feller</u> if

$S(t)$ maps $C_b(Y) \to C_b(Y)$ (= bounded

continuous functions on Y) $\Leftrightarrow$ $p(t, x, dy)$

is weakly continuous i.e. $\forall f \in C_b(Y)$

$x \to \int p(t, x, dy) f(y)$ is continuous.

<u>Remark</u> Some times $S(t): L^\infty(Y) \to C_b(Y)$

Then we say it is <u>strong Feller</u>. So

process <u>smoothens</u> (example: diffusion processes).

In our case there is no smoothing.

<u>Strong Markov Property</u>

R.V. $\tau: D_Y \to [0, \infty]$ is a stopping time if

$\{\omega \mid \tau(\omega) \leq t\} \in \mathcal{F}_t$ $\forall t < \infty$

Define the σ-algebra of events known at time τ

$$\mathcal{F}_\tau = \{A \in \mathcal{F} \mid A \cap \{\omega \mid \tau(\omega) \le t\} \in \mathcal{F}_t \quad \forall t < \infty\}$$

Def Θ_τ : $(\Theta_\tau \omega)(s) = \omega(\tau(\omega) + s)$

Th Let P^x be Feller. Then $\forall x$, $\forall f$ bounded

$$E^x(f \circ \Theta_\tau \mid \mathcal{F}_\tau)(\omega) = E^{\omega(\tau)}(f)$$

for P^x a.s. in ω s.t. $\tau(\omega) < \infty$

Proof 1. Let us first assume τ takes a countable set of values:

$$\tau(\omega) \in \{\tau_1, \tau_2, \dots\}.$$

Then, for $A \in \mathcal{F}_\tau$:

$$\circledast \quad E^x(f \circ \Theta_\tau \, \mathbb{1}_A \, \mathbb{1}_{\tau < \infty})$$

$$= \sum_{n=1}^{\infty} E^x(f \circ \Theta_{\tau_n} \mathbb{1}_A \mathbb{1}_{\tau = \tau_n})$$

Since $A \cap \{\omega \mid \tau(\omega) = \tau_n\} \in \mathcal{F}_{\tau_n}$ this equals

$$= \sum_{n=1}^{\infty} E^x(E^x(f \circ \Theta_{\tau_n} \mid \mathcal{F}_{\tau_n}) \mathbb{1}_A \mathbb{1}_{\tau = \tau_n})$$

$$\underset{\substack{\uparrow \\ \text{Markov prop}}}{=} \sum E^x((E^{X(\tau_n)} f) \mathbb{1}_A \mathbb{1}_{\tau = \tau_n})$$

$$\circledast \quad = E^x(E^{X(\tau)} f \, \mathbb{1}_A \mathbb{1}_{\tau < \infty}) \Rightarrow \text{Claim.}$$

[We just split into events according to value of τ and used Markov property].

2. Now lets prove $\circledast$ for f of form

$$f(\omega) = f_1(\omega(t_1))\, f_2(\omega(t_2)) \cdots f_n(\omega(t_n))$$

$$f_i \in C_b(Y) \qquad 0 \le t_1 < t_2 \cdots < t_n.$$

Define

$$\tau^{(n)} = 2^{-n}\left([2^n \tau] + 1 \right)$$

so $\quad \tau(\omega) \le \tau^{(n)}(\omega) \le \tau(\omega) + 2^{-n} \quad$ i.e.

$$\{\omega \mid \tau < \infty\} = \{\omega \mid \tau^{(n)} < \infty\}$$

$$\tau^{(n)} \downarrow \tau \qquad n \to \infty$$

$$\mathcal{F}_\tau \subseteq \mathcal{F}_{\tau_n}$$

$\tau^{(n)}$ takes countable # of values so

$$E^x\left(f \circ \theta_{\tau_n}\, \mathbb{1}_A\, \mathbb{1}_{\tau < \infty} \right) = E^x\left(f \circ \theta_{\tau_n}\, \mathbb{1}_A\, \mathbb{1}_{\tau^{(n)} < \infty} \right)$$

$$= E^x\left(E^{\omega(\tau^{(n)})} f \, \mathbb{1}_A\, \mathbb{1}_{\tau^{(n)} < \infty} \right)$$

$$= E^x\left(E^{\omega(\tau^{(n)})} f \, \mathbb{1}_A\, \mathbb{1}_{\tau < \infty} \right) \qquad (**)$$

$\forall A \in \mathcal{F}_\tau \quad (\subseteq \mathcal{F}_{\tau^{(n)}})$.

Now, for f of above form

$$E^x f = \int p(x, dx_1) \cdots p(x_{n-1}, dx_n) \prod f_i(x_i)$$

is continuous in x since $p(x, \cdot)$ is (show!)

$$\omega(\tau^{(n)}(\omega)) \xrightarrow[n \to \infty]{} \omega(\tau(\omega))$$

since $\tau^{(n)}(\omega) \downarrow \tau(\omega)$ and ω is right continuous. Hence can take $n \to \infty$ limit in $(**)$.

3. Finally prove $(*)$ for general $f = \mathbb{1}_B$

$B \in \mathcal{F}$ is by π-λ Theorem. $\square$

Poisson processes

Homogeneous Poisson process, rate r $\stackrel{\text{def}}{=}$ Markov process

N_t, state space $\mathbb{N}$, rate matrix

$$q(j, j+1) = r, \quad q(j,j) = -r \qquad r \in \mathbb{N}$$

i.e. generator

$$(Lf)(i) = \sum_j q(i,j)(f(j) - f(i))$$
$$= r(f(i+1) - f(i))$$

This means we can take $c(i) = r$, $p(i,j) = \delta_{j,i+1}$

i.e. the chain Y_i is deterministic $Y_i = i$

and times σ_i are i.i.d exponential mean $\frac{1}{r}$

$T_n = \sigma_0 + \sigma_1 + \dots + \sigma_{n-1}$. Take $N_0 = 0$

Then

$$N_t = n \quad \text{if } t \in [T_n, T_{n+1})$$

This equals also

$$N_t = \# \{ T_i : T_i \in (0, t] \}$$

i.e. $T_1 < T_2 < \dots$ are random points

on $\mathbb{R}$ and $N_t = \#$ points on $(0, t]$.

Define, for B a Borel set on $(0,\infty)$

$$N(B) = \#\{T_i \in B\} = \sum_{i=1}^{\infty} \mathbb{1}_B(T_i)$$

$N(\cdot)$ is a <u>random measure</u>

$$N = \sum_{i=1}^{\infty} \delta_{T_i} \qquad \delta_T \text{ point mass at } T$$

<u>Prop</u> Let $|B|$ be Lebesgue measure of B.

a) $N(B)$ is Poisson$(\lambda|B|)$ i.e.

$$P(N(B)=k) = \frac{e^{-\lambda|B|}(\lambda|B|)^k}{k!} \qquad k \in \mathbb{N}$$

and if $|B| = \infty$, $P(N(B)=\infty) = 1$.

b) If $B_1 \cap B_2 = \emptyset$ then $N(B_1)$ and $N(B_2)$ are independent.

<u>Proof</u> Let $f: \mathbb{R}^+ \to \mathbb{R}^+$ and

$$N(f) = \int f(x)\, N(dx) = \sum_{i=1}^{\infty} f(T_i)$$

Compute

$$E e^{-N(f)} = \lim_{n\to\infty} E e^{-\sum_{i=1}^{n} f(T_i)}$$

$$E e^{-\sum_{i=1}^{n} f(T_i)} = \int E\left(e^{-\sum_1^n f(T_i)} \mid T_{n+1} = t\right) P(T_{n+1} \in dt)$$

<u>Exercise</u>: Distribution of $(T_1, \dots, T_n)$ given $T_{n+1} = t$ equals distribution of $(S_1, \dots, S_n)$ where we take i.i.d $\tilde{S}_i$

$i=1,\dots n$, $\tilde{S}_i$ uniform on $[0,t]$ and

$\{S_i\}_{i=1}^n = \{\tilde{S}_i\}_{i=1}^n$ in order $S_1 < S_2 \dots < S_n$.

—o—

Hence

$$E\, e^{-N(f)} = \lim_{n\to\infty} \int_0^\infty \underbrace{E\, e^{-\sum_{i=1}^n f(S_i)}}_{=E\, e^{-\sum f(\tilde{S}_i)}} P(T_{n+1} \in ds)$$

$$= \lim_{n\to\infty} \int_0^\infty \left(E\, e^{-f(S)}\right)^n P(T_{n+1} \in ds)$$

$$= \lim \int_0^\infty \left(\int_0^s e^{-f(\sigma)} \frac{d\sigma}{s} \right)^n P(T_{n+1} \in ds)$$

$$= \lim_{n\to\infty} E\left(\int_0^{T_{n+1}} e^{-f(\sigma)} \frac{d\sigma}{T_{n+1}} \right)^n$$

$$= \lim_{n\to\infty} E\left(1 - \frac{\int_0^{T_{n+1}} (1-e^{-f(\sigma)})\, d\sigma}{T_{n+1}}\right)^{\frac{n}{T_{n+1}} \cdot T_{n+1}}$$

But $T_{n+1} = \sum_{i=0}^n \tau_i$; By Law of Large Numbers $\frac{T_{n+1}}{n} \xrightarrow[n\to\infty]{} 1/R$ a.s. Hence by Dominated convergence theorem

$$= \exp\left[R \int_0^\infty (e^{-f(\sigma)} - 1)\, d\sigma\right]$$

Take $f = \lambda \mathbb{1}_B$ $\Rightarrow$ get Laplace transform of Poisson distribution: $\exp[R|B|(e^{-\lambda}-1)]$ $\square$

So we view Poisson process as random set $\mathcal{T} = \{T_1, T_2, \ldots\}$.

Prop Let $(\mathcal{T}_i)$ be independent Poisson, rates r_i. Let $r = \sum r_i < \infty$, $\mathcal{T} = \bigcup_i \mathcal{T}_i$. Then $\mathcal{T}$ is Poisson rate r. For any $0 < s < \infty$ first point of $\mathcal{T}$ after s comes from $\mathcal{T}_i$ w. prob $\frac{r_i}{r}$.

Conversely. Let $\mathcal{T}$ Poisson r. Given $\mathcal{T}$ let $(Y_t)_{t\in\mathcal{T}}$ be i.i.d. $P(Y_t = i) = p_i$. Let $\mathcal{T}_i = \{t \in \mathcal{T} \mid Y_t = i\}$. Then $\mathcal{T}_i$ are independent Poisson rates $p_i r$

Pf Lets do the converse: Let $N_i = \sum_{t\in\mathcal{T}_i} \delta_t$

Compute

$$E e^{-\sum_i N_i(f_i)} = E e^{-\sum_{i=0}^{\infty} \sum_{t\in\mathcal{T}} f_i(t) \mathbb{1}_{Y_t = i}}$$

$$= E e^{-\sum_{t\in\mathcal{T}} f_{Y_t}(t)} = E_{\mathcal{T}}\left(E_Y e^{-\sum_{t\in\mathcal{T}} f_{Y_t}(t)} \mid \mathcal{T}\right))$$

$$\underset{\text{indep}}{=} E_{\mathcal{T}} \prod_{t\in\mathcal{T}} \underbrace{E_Y e^{-f_Y(t)}}_{= e^{-g(t)}} = E_{\mathcal{T}} e^{-\sum_{t\in\mathcal{T}} g(t)}$$

$$= E \exp[-N(g)] = \exp\left[r \int_0^\infty (e^{-g(t)} - 1)\, dt\right]$$

$$= \exp\left[r \int_0^\infty (E_Y e^{-f_Y(t)} - 1)\, dt\right]$$

$$= \exp\left[r \sum_{i=0}^{\infty} p_i \int_0^\infty (e^{-f_i(t)} - 1)\, dt\right]$$

$$= \prod_{i=0}^{\infty} \exp r p_i \int_0^\infty (e^{-f_i(t)} - 1)\, dt \qquad \Box$$

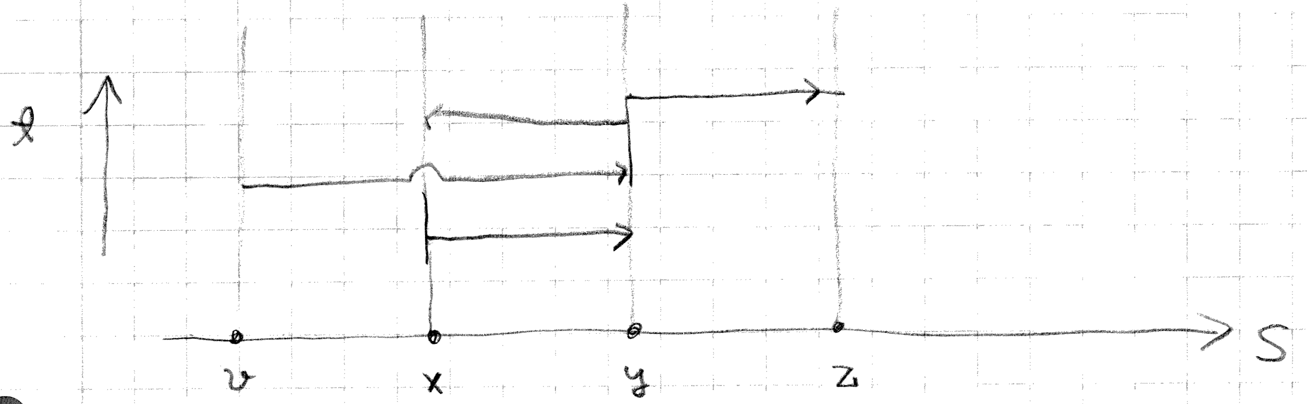
Graphical Representation of X_t^x

We defined X_t^x by

- Markov Chain $Y_0 = x, Y_1, \ldots$
- Waiting times $T_0, T_1, \ldots$ at Y_n

Let's give an x-independent construction.
State space S.

- put a Poisson clock on every possible jump $x \to y$:
  Let $x, y \in S$.
  $\mathcal{T}_{xy}$ = Poisson process, rate $q(x,y) = c(x)p(x,y)$
- $\mathcal{T} = \bigcup_{x,y \in S} \mathcal{T}_{xy}$.



Motion in $[0,\infty) \times S$: (t, X_t^x)

- move w. rate 1 in t
- jump when meet an arrow

X_t^x : $X_0^x = x \equiv y_0$

T_1 smallest time in $\bigcup_y \mathcal{T}_{y_0 y}$ i.e.

1st arrow from y_0 ; $y = y_1$

$X_{T_1}^x = y_1$

keep on going

$$\Rightarrow \quad 0 = T_0 < T_1 < T_2 \dots$$
$$x = y_0, y_1, y_2, \dots$$

Let us prove X_t^x defined on the prob. space $(\Omega, \mathcal{H}, P)$ of the Poisson $\mathcal{T}$ is a Markov process.

Let $\mathcal{H}_t$ = σ-algebra of $\mathcal{T}$ up to time t

Recall: May view $\mathcal{T}$ as • the random set $\subset [0,\infty)$
• counting function $N_t = | \mathcal{T} \cap [0,t] |$

Time shift:

$$(\theta_s N)_t = N_{t+s}$$

i.e. $(\theta_s N)(u,v] = N_{v+s} - N_{u+s} = N(s+u, s+v]$

i.e. $\theta_s \mathcal{T} \equiv \{ t-s \mid t \in \mathcal{T}, t > s \}$.

i.e. in $\theta_s N$ start counting at time s.

Write explicitely the $\mathcal{T}$ dependence of X_t^x:

$$X_t^x = G_t(x, \mathcal{T})$$

So $\quad p_t(x,y) = P(G_t(x,\mathcal{T}) = y)$

We show:

$$(*) = P(X_{s+t}^x = y \mid \mathcal{H}_s)(\omega) = p_t(X_t^x(\omega), y) \qquad (M)$$

Proof We have by construction:

$$X_{s+t}^x = G_t(X_s^x, \theta_s \mathcal{T}) \quad \text{so}$$

$$(*) = P(G_t(X_s^x, \theta_s\mathcal{T}) = y \mid \mathcal{H}_s)(\omega)$$

But: • X_s^x is $\mathcal{H}_s$ - measurable
• $\theta_s \mathcal{T}$ = depends on $\mathcal{T} \cap (s, \infty)$
= independent on $\mathcal{H}_s$ (Poisson on disjoint intervals are independent)

so

$$= P\left(G_t(z, \theta_s \mathcal{T}) = y \right) \Big|_{z = X_s^x(\omega)}$$

But $\theta_s \mathcal{T}$ has the same distribution as $\mathcal{T}$

$$= p_t(X_s^x(\omega), y) \qquad \square$$

Since $\{X_u^x \mid 0 \le u \le s\}$ is $\mathcal{H}_s$ measurable

(M) $\Rightarrow$ X_t^x is Markov with $p_t(x,y)$ Tr. prob.

Exclusion process

- At most one particle at $x \in \mathbb{Z}^d$
- attempt jump at rate 1
- jump $x \to y$ prob $p(x,y) = \tilde{p}(x-y)$ if y vacant
  if not, dont jump
- State space $\eta \in S = \{0,1\}^{\mathbb{Z}^d}$

1. Finite volume.

$$\eta \in \{0,1\}^{\mathbb{T}_N^d} = S$$

$(\mathcal{T}_{xy})_{x,y \in \mathbb{T}_N^d}$ independent Poisson, rate $p(x,y)$

$\mathcal{T} = \bigcup_{x,y} \mathcal{T}_{xy}$, $(\Omega, \mathcal{H}, \mathbb{P})$

$\mathcal{T}_{xy} = \{T_1 < T_2 \ldots\}$ attempted jump times for jump $x \to y$

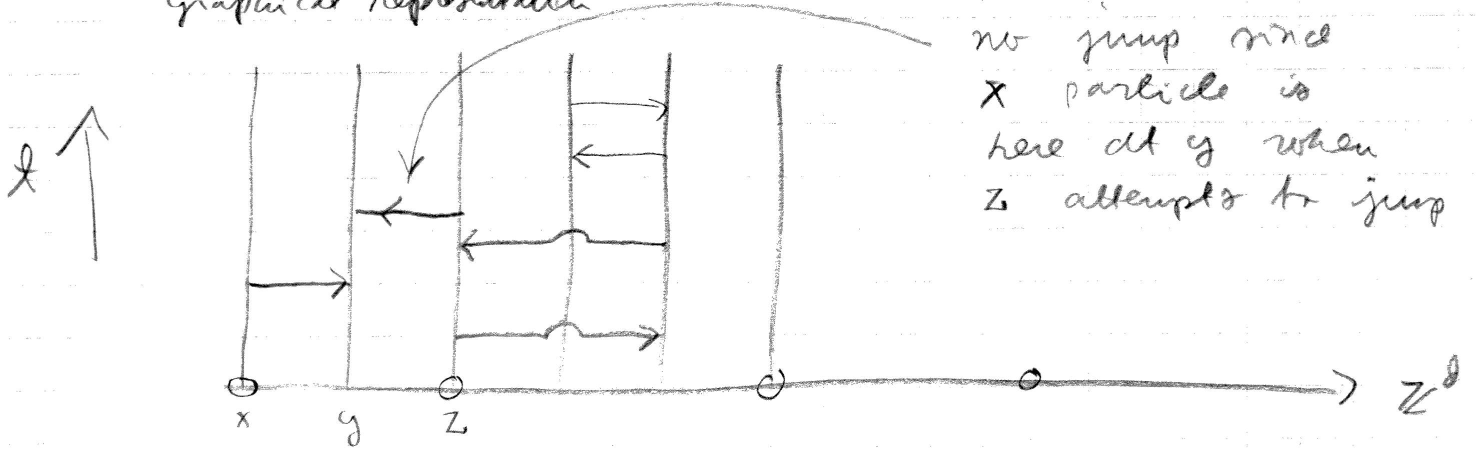
$\mathcal{T}_x = \bigcup_y \mathcal{T}_{xy}$ attempts $x \to$ anywhere

Suppose also $p(x,x) = \tilde{p}(0) = 0$. (no loss)

Clocks ringing for possible jumps
Actual jumps: if particle at x and y is vacant

$\mathbb{P}$(2 clocks ring at same time) $= 0$. So assume times separate.

Graphical representation



Let $0 < \tau_1 < \tau_2 \dots$ be the elements of $\mathcal{T}$

Let $\tau_i \in \mathcal{T}_{x_i y_i}$

Initial state $\eta_0 \in \{0,1\}^{\mathbb{T}_N^d}$

$$\eta_t = \eta_0 \quad t \in [0, \tau_1)$$

If $\eta_0(x_1) = 1$, $\eta_0(y_1) = 0$

$$\eta_{\tau_1} = \eta_0^{x_1 y_1}$$

$$\eta^{xy}(z) := \begin{cases} \eta(y) & z = x \\ \eta(x) & z = y \\ \eta(z) & z \neq x, y \end{cases}$$

If $\eta_0(x_1) = 0$ or $\eta_0(y_1) = 1$,

$$\eta_{\tau_1} = \eta_0$$

Keep on going!

Remark This construction is slightly different from our general Markov process (finite S) one: There Poisson processes were given for each $\underline{\eta, \eta' \in S}$ i.e. $\mathcal{T}_{\eta\eta'}$, rates $q(\eta, \eta')$

Here we give rates for each $\underline{x, x' \in \mathbb{T}_N^d}$ i.e. particle jumps not occupation # jumps.

One can show like there that one gets Markov process & compute the generator. We'll do this now directly in ∞ volume.

4

∞ volume

State space $Y = \{0,1\}^{\mathbb{Z}^d} \ni \eta = (\eta(x))_{x \in \mathbb{Z}^d}$

Probability space Ω: $\forall x, y \in \mathbb{Z}^d$ choose independent Poisson process $\mathcal{T}_{xy}$ on $[0,\infty)$ rate $p(x,y)$

Thus $\mathcal{T}_x = \bigcup_y \mathcal{T}_{xy}$ is Poisson rate

$$\sum_y p(x,y) = 1$$

In finite volume

$$\mathcal{T} = \bigcup_{x \in \mathbb{T}_N^d} \mathcal{T}_x \quad \text{has rate} \quad N^d < \infty$$

so we could list jump times $\tau \in T$ as $0 < \tau_1 < \tau_2 \dots$

In ∞ volume we can not! ∞ number of jumps in any time interval (measure N is not locally finite).

Way out: jumps far from x will not affect evolution of $\eta_t(x)$ for short times, i.e. there is a (probabilistically) finite propagation speed.

Random graph. Fix time t. Consider graph $\mathcal{G}_t$ on $\mathbb{Z}^d$ with edges

$$\mathcal{E}_t = \{\{x,y\},\ x,y\in\mathbb{Z}^d \mid (\mathcal{J}_{xy}\cup\mathcal{J}_{yx})\cap[0,t]\neq\phi\}$$

= set of pairs (x,y) s.t. $\exists$ possible jump $x\to y$

or $y\to x$ during time t.

Prop $\exists t_0>0$ s.t. almost surely all connected components of $\mathcal{G}_t$ are finite if $t\le t_0$. We suppose

p is <u>finite range</u> i.e. $p(x,y)=0$ if $|x-y|>r$.

Pf 1. a.s. the connected component of $\mathcal{G}_t$ containing 0 (origin) is finite if t small.

Pf Let $\mathcal{G}_t$ contain path $\gamma=\{\{0,x_1\},\{x_1,x_2\},\dots\{x_{n-1},x_n\}$

$$\text{Prob}(\gamma) = \prod_{i=1}^{n} 1-e^{-t(p(x_{i-1},x_i)+p(x_i,x_{i-1}))}$$

$$\le (1-e^{-2t})^n$$

$$\#\text{ of such }\gamma \le [(cr)^d]^n$$

$$\Rightarrow \text{Prob}(\underbrace{\exists\text{ path, length } n}_{A_n(\omega)}) \le [c(1-e^{-2t})r^d]^n$$

$$< \lambda^n \qquad \lambda<1 \text{ if } t<t_0(r)$$

$$\Rightarrow \sum_{n=1}^{\infty} \text{—''—} < \infty \qquad t<t_0$$

Borel Cantelli: $P(A_n$ happens ∞ often$)=0$

i.e. a.s. $n<\infty$ i.e. a.s.

there is R s.t. 0 not connected to B_R^c.

2. Translation invariance ($p(x,y)=\tilde p(x-y)$) $\Rightarrow$

$P($ $x \in$ infinite connected component of $\mathcal{G}_\ell)$

is x independent $= 0 \Rightarrow$

$P(\exists \infty$ c.c.$) = 0$. $\square$

Let $\tilde{\Omega}_0 = \{\omega \in \Omega \mid$ all jump times in J_{xy}, $x, y \in \mathbb{Z}^d$

are different and all c.c. of $\mathcal{G}_{\ell_0}$ are finite $\}$

Then $P(\tilde{\Omega}_0) = 1$. For $\omega \in \tilde{\Omega}_0$ we can

construct $\eta_\ell(x, \omega)$ as in the $\mathbb{T}_N^d$ case:

in each c.c. list jump times $0 < \tau_1 < \tau_2 \dots$

and do as before. $\Rightarrow$ $\eta_\ell(x, \omega)$ are defined

$\ell \le \ell_0$, $x \in \mathbb{Z}^d$, $\omega \in \tilde{\Omega}_0$.

Now, on $(\ell_0, 2\ell_0]$ the c.c. of graph generated

from $\bigcup_{xy} J_{xy} \cap (\ell_0, 2\ell_0]$ are finite on

$\omega \in \tilde{\Omega}_1$, $P(\tilde{\Omega}_1) = 1$. Now, get η up to $2\ell_0$

Keep on going, in

$$\omega \in \tilde{\Omega} = \bigcap_{n=0}^{\infty} \Omega_n$$

get $\eta_\ell(x, \omega)$ for all x, ℓ and $P(\tilde{\Omega}) = 1$.

Markov property

We constructed $\eta_\ell(x, \omega)$ $\omega \in \tilde{\Omega} \subset \Omega$ where

$$\Omega = \bigtimes_{(x,y) \in \mathbb{Z}^d} \Omega_{xy}$$

Ω_{xy} prob. space where Poisson process

$N_{xy}(t) = J_{xy} \cap (0,t]$ is defined.

Recall that we can view each of these as defined on path spaces $D_{\mathbb{N}}[0,\infty)$ of RCLL paths $t \to N_{xy}(t)$.

We can view all of them at once:

State space $U = \mathbb{N}^{\mathbb{Z}^d \times \mathbb{Z}^d}$ i.e. $N \in U$ is $N = \{N_{xy}\}_{x,y \in \mathbb{Z}^d}$

Metric on U

$$d(N,N') = \sum_{x,y} 2^{-|x|-|y|} \gamma(N_{xy}, N'_{xy})$$

γ metric on $\mathbb{N}$: $\gamma(n,m) = \min\{|n-m|, 1\}$.

So, N, N' are close if $N_{xy} = N'_{xy}$ in a large nhood of origin.

The path $\omega(t) = \{N_{xy}(t)\}_{x,y \in \mathbb{Z}^d} \in U$ is RCLL in $D_U[0,\infty)$, which is Polish.

Two paths ω, ω' are close if in a large box B around origin and up to large time T

$$N_{xy}(t) = N'_{xy}(\lambda(t)) \qquad \forall x,y \in B, \ t \le T$$

and $\lambda : [0,T] \to [0,\infty)$ close to identity map.

Denote $\eta_t^{\eta}(x,\omega)$ path with $\eta_0 = \eta$.

We have

1. $t \to \eta_t^\eta(\omega)$ is RCLL $(\omega \in \tilde{\Omega})$

2. $(\eta, \omega) \to \eta_\cdot^\eta(\omega)$ is continous map

$U \times \tilde{\Omega} \to D_X[0,\infty)$ $\quad X = \{0,1\}^{\mathbb{Z}^d}$

Pf 1. RC: Let B_R = ball radius R in $\mathbb{Z}^d$

jump times in B, during (t, T) finite $\forall T<\infty$

$\Rightarrow \exists \delta > 0$: no jumps in B during $(t, t+\delta)$

$\Rightarrow \eta_t^\eta(x,\omega) = \eta_{t+s}^\eta(x,\omega) \quad \forall s \in (0,\delta)$
$x \in B$

so $d(\eta_t^\eta(\omega), \eta_{t+s}^\eta(\omega)) \leq \sum_{x \in B^c} 2^{-|x|}$

$\leq \sum_{n=R}^{\infty} 2^{-n} c n^d \to 0$ as $R \to \infty$. I.e.

$\forall \varepsilon \exists \delta > 0$: $\sup_{s \leq \delta} d(\eta_t^\eta(\omega), \eta_{t+s}^\eta(\omega)) \leq \varepsilon \Rightarrow \lim_{s \downarrow 0} \eta_{t+s}^\eta(\omega) = \eta_t^\eta(\omega)$ □

LL Fix t. $\eta_t(x)$ might jump at t but

$\forall x \, \exists \delta_x \; \bigcup_y (J_{yx} \cup J_{xy}) \cap (t-\delta_x, t) = \emptyset$

$\Rightarrow \eta_s(x,\omega)$ constant on $(t-\delta_x, t]$

$\Rightarrow \exists \lim_{s \uparrow t} \eta_s(x,\omega)$. $\Rightarrow \exists \lim \eta_s(\omega)$ □
Product topology

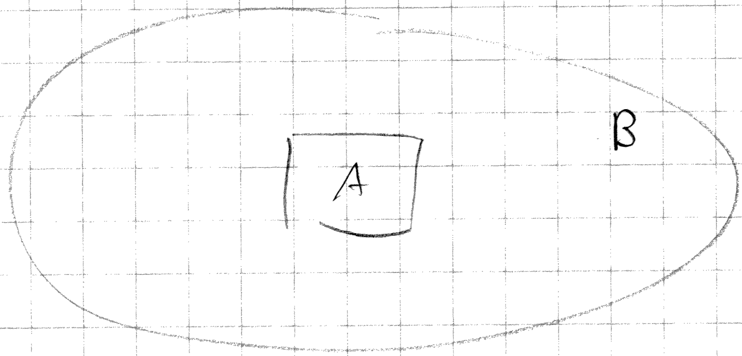
2. Fix η, ω.

A) Let A be arbitrary cube, $T < \infty$

Then $\exists B(A,T) \subset \mathbb{Z}^d$ s.t.

if $\eta(x) = \eta'(x) \quad \forall x \in B$

then $\eta_t^{\eta}(x) = \eta_t^{\eta'}(x) \quad \forall x \in A, \; t \le T$



Proof. Just add c.c. of the graphs $\mathcal{G}_{[nt_0,(n+1)t_0]}$

$$B_1 = \bigcup_{x \in A} \underbrace{C_{[0,t_0]}(x)}_{\text{c.c. of } \mathcal{G}_{[0,t_0]}(x)}$$

$$B_2 = \bigcup_{x \in B_1} C_{[t_0,2t_0]}(x) \quad \text{etc} \quad \square.$$

Let ρ be the metric in $D_U[0,\infty)$.

Given $\varepsilon > 0$ we can find $\delta > 0$ s.t. if

$\rho(\omega,\omega') < \delta$ then $\gamma(\lambda) = \sup_{s,t \le T} \left| \frac{\lambda(s)-\lambda(t)}{s-t} \right| < \varepsilon$,

$$N_{xy}(t) = N'_{xy}(\lambda(t)) \qquad 0 \le t \le T \qquad (*)$$

$\forall x,y$ s.t. $x \in B$ or $y \in B$

Taking δ smaller can assume $\eta(x) = \eta'(x) \quad \forall x \in B$.

So $\forall x \in A, \; t \le T$

$$\eta_t^{\eta}(x,\omega) = \eta_t^{\eta'}(x,\omega) \underset{\text{By } (*)}{=} \eta_t^{\eta'}(x,\omega' \circ \lambda) = \eta_{\lambda(t)}^{\eta'}(x,\omega')$$

Thus

$$\sup_{0 \le t \le T} d(\eta^{\eta}_t(\omega), \eta^{\eta'}_{t(t)}(\omega'))$$

$$\le \sum_{x \notin A} 2^{-|x|} \to 0 \quad \text{as } A \uparrow \mathbb{Z}^d.$$

i.e. can find $\delta > 0$, $d(\eta, \eta') < \delta$, $\rho(\omega', \omega) < \delta$

$$\Rightarrow \quad \rho(\eta^{\eta}(\omega), \eta^{\eta'}(\omega')) < \varepsilon \qquad \square$$

<u>Remark</u> $\tilde{\Omega}$ is actually open set (why?)

so $\eta^{\eta}_{\cdot} : \tilde{\Omega} \to D_X[0, \infty)$ continous and

so measurable.

We can now define prob measure P^{η}

on $(D_X, \mathcal{F})$ by

$$P^{\eta}(A) = P(\{\omega \mid \eta^{\eta}_{\cdot}(\omega) \in A\})$$

Then:

<u>Prop</u> $\{P^{\eta}\}$ is a Markov process

<u>Proof</u> 1. Clearly $P^{\eta}(\eta_{t=0} = \eta) = 1$

2. $\eta \to P^{\eta}(A)$ is measurable follows

easily from $\eta \to \eta^{\eta}_{\cdot}(\omega)$ continous so measurable

by looking at A = cylinder set

3. Markov property $E^{\eta}(f \circ \theta_t \mid \mathcal{F}_t) = E^{\eta_t}(f)$

follows exactly as in our S = countable case.

<u>Prop</u> Exclusion process is Feller.

<u>Pf</u> Note: $X = \{0,1\}^{\mathbb{Z}^d}$ is compact. Thus $C_b(X) = C(X)$

(continuous functions are bounded).

Show: $E^\eta(f(\eta_t))$ continuous in η if $f \in C(X)$.

$$= \underset{\substack{\uparrow \\ \text{in Poisson}}}{\mathbb{E}} \left(f(\eta_t^\eta) \right) \qquad (*)$$

But $|f(\eta_t^\eta(\omega))| \leq \sup|f(\eta)| < \infty$

$\eta \to f(\eta_t^\eta(\omega))$ continuous

$\Rightarrow$ $(*)$ is $\square$

<u>Cor</u> Strong Markov holds

<u>Generator</u>

Semigroup $(S(t)f)(\eta) = E^\eta(f(\eta_t))$

Lets look at this as $t \to 0$. Need to pay attention to ∞ volume, the generator <u>will</u> <u>not</u> be defined for all $f \in C(X)$.

<u>1. Finite volume</u>

Let $\eta \in \{0,1\}^{\mathbb{T}_N^d}$. We can proceed in a brutal* way as we did in $S =$ countable case. Let J be the combined Poisson process $\bigcup_{x,y \in \mathbb{T}_N^d} J_{xy}$. J has rate

* and stupid!

$\beta = \sum_{x,y \in \mathbb{T}_N^d} p(x,y)$. Let N_t be as usual

$|\mathcal{J} \cap [0,t]|$ i.e. # of jumps on $(0,t]$.

Then

$$(S(t)f)(\eta) = E^\eta f(\eta_t) = \underbrace{E^\eta(f(\eta_t)\, \mathbb{1}_{N_t=0})}_{(1)}$$

$$+ \underbrace{E^\eta(f(\eta_t)\, \mathbb{1}_{N_t=1})}_{(2)} + \underbrace{E^\eta(f(\eta_t)\, \mathbb{1}_{N_t>1})}_{(3)}$$

$$(1) = f(\eta)\, e^{-\beta t} = (1 - \beta t + O(\beta^2 t^2))\, f(\eta) = (1 - \sum p(x,y)\, t)\, f(\eta) + O(\beta^2 t^2)$$

$$(2) = \underbrace{E^\eta(f(\eta_t) \mid N_t = 1)}_{(*)}\ \underbrace{P(N_t = 1)}_{= \beta t e^{-\beta t}}$$

$$(*) = E_{\mathcal{J}}(f(\eta^\eta(\omega(t))) \mid N_t = 1) = \sum_{x,y \in \mathbb{T}_N^d} \frac{p(x,y)}{\beta} \cdot \begin{cases} f(\eta^{xy}) & \eta(x)=1,\ \eta(y)=0 \\ f(\eta) & \text{otherwise} \end{cases}$$

where $\frac{p(x,y)}{\beta}$ is the probability that the

jump is in $\mathcal{J}_{xy}$ and η^{xy} is as before.

(3) is $\leq \|f\|_\infty\, P(\text{more than one jump})$

$$\leq C \|f\|_\infty\, \beta^2 t^2$$

Then $(S(t)f)(\eta) - f(\eta) = t \sum_{xy} p(x,y) \left(-f(\eta) + \begin{cases} f(\eta^{xy}) & \eta(x)=1,\ \eta(y)=0 \\ f(\eta) & \text{other} \end{cases} \right)$

$$+ O\, \|f\|_\infty\, \beta^2 t^2$$

$$= (Lf)(\eta)\, t + O(\|f\|_\infty\, \beta^2 t^2)$$

$$(Lf)(\eta) = \sum_{x,y} p(x,y)\,\eta(x)(1-\eta(y))\,(f(\eta^{xy}) - f(\eta))$$

So $\lim_{t\to 0} \frac{S(t)f - f}{t} = Lf$.

<u>Note</u> This breaks down as $N \to \infty$! We got

$$\left\| \frac{S(t)f - f}{t} \right\|_\infty \le C\beta^2 t \, \|f\|_\infty$$

when $\beta = \sum_{x,y \in \mathbb{T}_N^d} p(x,y) = N^d \xrightarrow[N\to\infty]{} \infty$.

Need to be more clever in ∞ volume!

2. Infinite volume

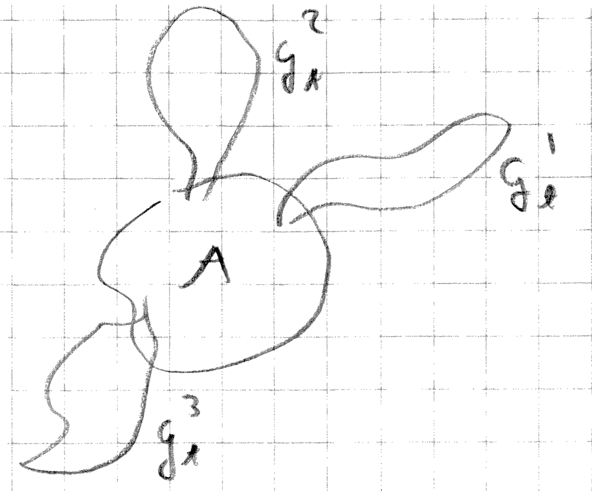
Take f <u>local</u> i.e. depending on finitely many $\eta(x)$'s (i.e. a cylinder function)

$$f(\eta) = \tilde{f}(\eta(x_1), \dots, \eta(x_m))$$

Call $A = \{x_1, \dots, x_m\} \subset \mathbb{Z}^d$ the <u>support of f</u>.

Recall $\mathcal{G}_t(\omega)$, the random graph formed by edges $\{x,y\}$ s.t. $\mathcal{T}_{xy} \cap [0,t] \neq \emptyset$.

For $t \le t_0$ and $\omega \in \tilde{\Omega}$, c.c. of $\mathcal{G}_t(\omega)$ are finite. Let C_t = union of —"— intersecting A



$= C_t$

Then $f(\eta(t))$, $t \leq t_0$ depends only on

$$\{\eta(x) \mid x \in C_{t_0}\} \quad \text{and} \quad \bigcup_{\{x,y\} \cap C_{t_0} \neq \emptyset} J_{xy}$$

Recall: r is range of p; $p(x,y) = 0$ if $|x-y| > r$.

Let
$$A_r = \{x \in \mathbb{Z}^d \mid \mathrm{dist}(x, A) > 2r\}$$

Thus, we need at least 3 jumps to go from A to A_r^c. Let

$$H_t = \{\omega \mid C_t \subseteq A_r\}$$

Now we repeat finite volume analysis with

$$N_t = \Big|\Big(\bigcup_{\{x,y\} \cap A_r \neq \emptyset} J_{xy}\Big) \cap (0,t]\Big|$$

the jumps where either x or y are in A_r.

Note: $\{\omega : N_t = 0\}$ and $\{\omega : N_t = 1\}$ $\subset H_t$ since only one jump intersecting A_r can not connect A to A_r^c. So

$$\mathbb{1} = \mathbb{1}_{N_t = 0} + \mathbb{1}_{N_t = 1} + \mathbb{1}_{H_t} \mathbb{1}_{N_t > 1} + \mathbb{1}_{H_t^c}$$

$$(S(t)f)(\eta) - f(\eta) = \underbrace{E\left(f(\eta_t^\eta) - f(\eta) \right) \mathbb{1}_{N_t = 0}}_{(1)}$$

$$+ \underbrace{E\left(f(\eta_t^\eta) - f(\eta) \right) \mathbb{1}_{N_t = 1}}_{(2)} + \dots \underbrace{\mathbb{1}_{H_t} \mathbb{1}_{N_t \geq 1}}_{(3)} + \dots \underbrace{\mathbb{1}_{H_t^c}}_{(4)}$$

(1) = 0 since no jump

$(2) = t e^{-\beta t} (Lf)(\eta)$

$|(3)| \leq 2\|f\|_\infty \, P(N_t > 1) \leq 2\beta^2 t^2 \|f\|_\infty$

and for (4) there has to be a connected component of the graph $\mathcal{G}_t$ of length 3 or more edges intersecting A.

Probability of this is

$$\leq (\#\text{ points in } A) * \sum_{n=3}^{\infty} \left(c R^d (1 - e^{-2t}) \right)^n$$

$$\leq C(R) |A| t^3$$

Since β = rate of N_t $= \sum_{\substack{x \text{ or } y \\ \text{in } A_R}} p(x,y) \leq |A_R|$

we got

$$\left\| \frac{S_t f(\eta) - f(\eta)}{t} - Lf(\eta) \right\| \leq t \|f\|_\infty C(A, R)$$

where $C \sim |A|^2$ as $|A| \to \infty$.

We got:

Prop For f cylinder function $\frac{d}{dt}\Big|_{t=0} S(t) f(\eta)$ exists and equals $(Lf)(\eta)$. In fact

$$\lim_{t\to 0} \sup_{\eta} \left[\frac{1}{t} (S(t) f(\eta) - f(\eta)) - Lf(\eta) \right] = 0$$

A martingale

In finite volume $\mathbb{T}_N^d$ we have for all $f \in C(X_N)$

$$\frac{d}{dt}\Big|_0 S(t) f(\eta) = Lf(\eta) \qquad (X_N = \{0,1\}^{\mathbb{T}_N^d})$$

Thus, $S(t) S(s) = S(t+s) \Rightarrow \frac{d}{dt} S(t) f(\eta) = S(t) Lf(\eta)$

and $S(t) f(\eta) = E^\eta f(\eta_t) \Rightarrow$

$$E^\eta f(\eta_t) - E^\eta f(\eta_s) = \int_s^t E^\eta (Lf)(\eta_u) du$$

$$= E^\eta \int_s^t (Lf)(\eta_u) du \qquad (*)$$

Define

$$M_t = f(\eta_t) - \int_0^t (Lf)(\eta_u) du$$

Then, M_t is a martingale w.r.t. to the filtration $\mathcal{F}_t$ i.e.

$$E^\eta (M_t \mid \mathcal{F}_s) = M_s$$

Indeed, $M_t - M_s = f(\eta_t) - f(\eta_s) - \int_s^t (Lf)(\eta_u) du$

$$E^\eta (M_t - M_s \mid \mathcal{F}_s) = E^\eta (f(\eta_t) \mid \mathcal{F}_s) - f(\eta_s)$$

$$- \int_s^t E^\eta (Lf(\eta_u) \mid \eta_s) du \underset{\text{Markov}}{=}$$

$$= E^{\eta_s} f(\eta_{t-s}) - f(\eta_s) - \int_s^t E^{\eta_s} Lf(\eta_{u-s})\,du$$

$$\overset{*}{=} f(\eta_s) + \int_0^{t-s} E^{\eta_s}(Lf)(\eta_u)\,du - f(\eta_s) - \int_0^{t-s} E^{\eta_s} Lf(\eta_u)\,du$$

$= 0$ □

<u>Example</u> Consider the <u>symmetric</u> exclusion process i.e. $p(x,y) = p(y,x)$. Then symmetrize:

$$Lf(\eta) = \frac{1}{2} \sum_{x,y} [\eta(x)(1-\eta(y)) + \eta(y)(1-\eta(x))]\, p(x,y) \cdot [f(\eta^{xy}) - f(\eta)]$$

Now $f(\eta^{xy}) - f(\eta) = 0$ unless $\eta(x) \neq \eta(y)$

But then $\eta(x)(1-\eta(y)) + \eta(y)(1-\eta(x)) = 1$ so

$$Lf(\eta) = \frac{1}{2} \sum_{x,y} p(x,y)\,[f(\eta^{xy}) - f(\eta)]$$

Take next a special f, and consider <u>finite volume</u> $\mathbb{T}_N^d$:

$$f(\eta) = \sum_{x \in \mathbb{T}_N^d} \eta(x)\, g(x)$$

Get $$Lf(\eta) = \frac{1}{2} \sum_{u,v,x} p(u,v) \left(\eta^{uv}(x) - \eta(x)\right) g(x)$$

so $\neq 0$ only if $u = x$ or $v = x$

$$= \frac{1}{2} \sum_{x,v} p(x,v)\,(\eta(v) - \eta(x))\, g(x)$$

$$= \frac{1}{2} \sum_{x,v} p(x,v)\, \eta(x)\,(g(v) - g(x))$$

$$= \frac{1}{2} \sum \eta(x) (\tilde{\Delta} g)(x)$$

$$\tilde{\Delta} g(x) = \left(\sum_v p(x,v)\, g(v)\right) - g(x)$$

Lets define "macroscopic density"

$$\rho_N(t, dx) = \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} \eta_{N^2 t}(x)\, \delta_{x/N}$$

a measure on $\mathbb{T}^d$. So, for $g \in C^\infty(\mathbb{T}^d)$

$$\rho_N(t, g) = \frac{1}{N^d} \sum_x g(x/N)\, \eta_{N^2 t}(x)$$

We got

$$\rho_N(t, g) - \rho_N(0, g) = \int_0^t ds\, \rho_N(s, \Delta_N g) + M_{N,t}$$

where $M_{N,t}$ is a martingale and

$$(\Delta_N g)(x) = N^2\Big(\sum_{v \in \mathbb{T}_N^d} p(Nx, v)\, g(v/N) - g(x)\Big)$$

(N^2 is from change of variables in time integral)

No $p(x,y) = \tilde{p}(x-y) = p(y,x) = \tilde{p}(y-x)$ so

$$\sum_x \tilde{p}(x) = 1 \qquad \sum_x \tilde{p}(x) x = 0$$

Let

$$D_{ij} = \sum_x \tilde{p}(x)\, x_i x_j$$

Then (prove!)

$$(\Delta_N g)(x) \to \sum_{ij} D_{ij}\, \partial_i \partial_j g(x)$$

If we could show $M_{N,t} \xrightarrow[N\to\infty]{} 0$ then we might be able to prove

$$\rho_N(t, dx) \xrightarrow[N\to\infty]{} \rho(t, x)\, dx$$

$$\partial_t \rho(t,x) = D_{ij}\,\partial_i \partial_j\, \rho(t,x)$$

This will crucially depend on the initial measure of η. Should take a "local equilibrium measure.

Semigroups & generators

In infinite volume L is not defined on all $f \in C(X)$ so need to be careful with above formuli.
Let B be Banach space. For us $B = C(X)$, X compact metric.
Let $S(t) : B \to B$ be contractive semigroup

$$S(t+s) = S(t)S(s)$$
$$\| S(t) f \|_\infty \le \|f\|$$

Suppose also S is strongly continuous

$$\| S(t) f - f \| \xrightarrow[t\to 0]{} 0 \qquad \forall f \in B \qquad (*)$$

Example Exclusion process, $B = C(X)$, $X = \{0,1\}^{\mathbb{Z}^d}$, is.

Proof We have shown $(*)$ (and more!) on cylinder functions $f \in \Sigma$. But these are dense in $C(X)$: $\forall g \in C(X)$ $\forall \varepsilon > 0$ $\exists f \in \Sigma$, $\|f - g\| < \varepsilon$.

Proof $C(X)$ is compact $\Rightarrow$ g is uniformly continuous: $|g(\eta) - g(\eta')| < \varepsilon$ if $d(\eta, \eta') < \delta$.
Pick $V \subset \mathbb{Z}^d$ finite s.t. $d(\eta, \eta_V) < \delta$ $\forall \eta$

$\eta_V = \begin{cases} \eta & \text{on } V \\ 0 & \text{on } V^c \end{cases}$. Put $f(\eta) = g(\eta_V)$ = cylinder. $\square$

So (*) holds on a dense set of f $\Rightarrow$ holds for all (Why?). $\square$

Define $\quad \mathcal{D}(L) := \{ f \mid \exists \lim_{t\to 0} \frac{S(t)f - f}{t} = g \in \mathcal{B} \}$

and $\quad L : \mathcal{D}(L) \to \mathcal{B}$

$$Lf = \lim \frac{S(t)f - f}{t}$$

L is the generator and $\mathcal{D}(L)$ its domain.

= linear subspace of $\mathcal{B}$

In finite volume exclusion $\mathcal{D}(L) = C(X)$, as volume $\neq \infty$.

Lemma a) $S(t)f$ is uniformly continuous in $t \in [0,\infty)$

b) $\forall f \in \mathcal{B}: \int_0^t ds\, S(s)f \in \mathcal{D}(L)$ & $S(t)f - f = L\int_0^t S(s)f\, ds$

c) If $f \in \mathcal{D}(L)$ then $S(t)f \in \mathcal{D}(L)$ and

$$\frac{d}{dt} S(t)f = LS(t)f = S(t)Lf$$

d) $$S(t)f - f = \int_0^t LS(s)f\, ds = \int_0^t S(s)Lf\, ds \quad \forall f \in \mathcal{D}(L),\ t \geq 0$$

Pf a) $\|S(t+s)f - S(t)f\| = \|S(t)(S(s)f - f)\| \leq \|S(s)f - f\|$ $\square$

b) Set $g = \int_0^t ds\, S(s)f$.

$$S(\tau)g = S(\tau)\int_0^t ds\, S(s)f \underset{S(\tau)\text{ bounded}}{=} \int_0^t ds\, S(\tau)S(s)f$$

$$= \int_0^t ds\, S(\tau+s)f$$

so $$\frac{S(\tau)g - g}{\tau} = \frac{1}{\tau}\int_\tau^{t+\tau} S(s)f\, ds - \frac{1}{\tau}\int_0^\tau S(s)f\, ds$$

$$\xrightarrow[(a)]{} S(t)f - f \quad \text{as } \tau \to 0 \quad \square$$

c) Use

$$\frac{1}{s}\left(S(t+s)f - S(t)f\right) = \underbrace{\frac{1}{s}(S(s)-1)\,S(t)f}_{*} = \underbrace{S(t)\,\underbrace{\frac{S(s)f-f}{s}}_{\to Lf}}_{\to S(t)Lf \text{ since } S(t) \text{ continuous}}$$

$\Rightarrow$ (*) converges

$\Rightarrow$ $S(t)f \in \mathcal{D}(L)$, $L\,S(t)f = S(t)Lf$

$\Rightarrow$ LHS converges $\Rightarrow$ $S(t)f$ diff from right

From left, similarly...

d) Follows by integrating since $S(t)Lf$ is continuous in t □

Corollary Let η_t be Markov, $S(t)$ Feller on $C_0(X)$. Then

$$M_t = f(\eta_t) - f(\eta_0) - \int_0^t (Lf)(\eta_s)\,ds \qquad f \in \mathcal{D}(L)$$

is a right continuous martingale, $EM_t = 0$.

Pf Need to show

a) M_t is $\mathcal{F}_t$-measurable

b) $E(M_t \mid \mathcal{F}_s) = M_s \quad s<t$.

a) follows by writing

$$\int_0^t (Lf)(\eta_s)\,ds = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (Lf)(\eta_{it/n})$$

b) is the same calculation as for L bounded using Lemma □

In case generator L of Markov is bounded (like exclusion in $\mathbb{T}_N^d$), L determines $S(t)$:

$$S(t) = e^{tL} = \sum_{n=0}^{\infty} \frac{t^n}{n!} L^n$$

converges ($\|L^n\| \le \|L\|^n$).

What of this remains if $\mathcal{D}(L) \neq C(X)$?

We will show that it suffices to work with cylinder functions.

Some general theory:

Def. $\mathcal{D}(A) \subset B$ subspace, $A : \mathcal{D}(A) \to B$ linear operator. A is <u>closed</u> if

$$\text{Graph}(A) : \{ (f, Af) \mid f \in \mathcal{D}(A) \}$$

is a closed subspace of $B \oplus B$ i.e.

if $f_n \to f$, $Af_n \to g$ then $f \in \mathcal{D}(A)$ and $g = Af$.

<u>Prop</u> If L is generator of strongly continuous semigroup then $\mathcal{D}(L)$ is dense and L is closed.

<u>Pf</u> By (A) of Lemma

$$\frac{1}{t} \int_0^t S(s) f \, ds \in \mathcal{D}(L) \quad \forall f \in B, \ t > 0.$$

$s \to S(s)f$ continuous $\Rightarrow$ $\xrightarrow[t \to 0]{} f$ so $\mathcal{D}(L)$ dense.

Suppose $f_n \to f$, $Lf_n \to g$, $f_n \in \mathcal{D}(L)$.

Then from Lemma d)

$$S(t)f_n - f_n = \int_0^t S(s) L f_n \, ds$$

Now RHS $\to \int_0^t S(s) g \, ds$:

$$\left\| \int_0^t S(s) L f_n \, ds - \int_0^t S(s) g \, ds \right\| \le \int_0^t \| S(s)(L f_n - g) \| \, ds$$

$$\le \int_0^t \| L f_n - g \| \, ds \le t \| L f_n - g \|$$

contract.

$\Rightarrow \quad S(t)f - f = \int_0^t S(s) g \, ds$ so

$$\frac{S(t)f - f}{t} \xrightarrow[t \to 0]{} g \quad \text{so} \quad f \in \mathcal{D}(L), \; g = Lf \quad \square$$

__Def__ Subspace $V \subset \mathcal{D}(L)$ is a __core__ for L if:

$$\forall f \in \mathcal{D}(L) \; \exists f_n \in V \text{ s.t. } f_n \to f, \; L f_n \to L f$$

(i.e. if $\overline{\text{graph of } L\restriction_V} = \text{graph of } L$)

We need:

__Prop__ Suppose V_0, V_1 are dense subspaces $V_0 \subset V_1 \subset \mathcal{D}(L)$. Suppose

$$S(t) : V_0 \to V_1 \quad \forall t \ge 0.$$

Then V_1 is a core for L.

__Pf__ See Seppäläinen notes $\square$