9.1. Suppose:
- a bird of type i lays n_i eggs
- probability of surviving first year of life s is a decreasing function of n.
- assume birds get adult in age 1 and survive then each year with probability p, which also decreases with n.
- assume s also depends on N (the total population density)

We have population dynamics:

$$N_i(t+1) = [n_i\, s(n_i, N(t)) + p(n_i)]N_i(t)$$

a.) Show: Generically only one type can be present in equilibrium, i.e. there exists an optimal fecundity n^* and it maximizes the equilibrium population density.

Assume type 1 is at equilibrium, i.e. $N_1(t+1) = N(t) = \hat{N}_1$

$$1 = n_1\, s(n_1, \hat{N}_1) + p(n_1)$$

$$\Rightarrow \quad s(n_1, \hat{N}_1) = \frac{1-p(n_1)}{n_1}$$

If type 2 was at equilibrium (alone)

$$s(n_2, \hat{N}_2) = \frac{1-p(n_2)}{n_2}.$$

Assume 1 is now at equilibrium. Type 2 can then invade, if:

$$n_2\, s(n_2, \hat{N}_1) + p(n_2) > 1$$

$$\Rightarrow \quad s(n_2, \hat{N}_1) > \frac{1-p(n_2)}{n_2} \overset{!}{=} s(n_2, \hat{N}_2) \qquad (*)$$

since s decreases with $\hat{N}$, $\hat{N}_1 < \hat{N}_2$.

From this it also follows that type 1 cannot invade back: If it could, we would have

$$n_1\, s(n_1, \hat{N}_2) + p(n_1) > 1$$

$$s(n_1, \hat{N}_2) > s(n_1, \hat{N}_1),$$ which contradicts with $(*)$

$\longrightarrow$

Coexistence of types would be possible if $\hat{N}_1 = \hat{N}_2$, which generically is not true.

b.) suppose juvenile survival is of the form

$$s(n, N) = s(n) f(N), \quad \text{where} \quad f(0) = 1.$$

Show that optimal fecundity n^* also maximizes function $R_0(n) = \dfrac{n\, s(n)}{1 - p(n)}$

assume again type 1 at equilibrium. Type 2 can invade if:

$$n_2 s(n_2) f(\hat{N}_1) + p(n_2) > 1$$

$$\underbrace{f(\hat{N}_1)}_{=} > \frac{1 - p(n_2)}{n_2\, s(n_2)}$$

$$\Rightarrow \frac{1 - p(n_1)}{n_1 s(n_1)} > \frac{1 - p(n_2)}{n_2 s(n_2)}$$

$$\Rightarrow \frac{n_1\, s(n_1)}{1 - p(n_1)} < \frac{n_2\, s(n_2)}{1 - p(n_2)} = R_0(n_2)$$

Evolution goes towards larger values of R_0.

9.2 $\dot{R} = rR(1-\frac{R}{K}) - \sum_i \beta_i N_i R$

$\dot{N}_i = (\gamma\beta_i R - \mu_i - b_i P) N_i$

$\dot{P} = (\sum_i c\, b_i N_i - \delta) P$

(a) For coexistence, $\gamma\beta_i R - b_i P = \mu_i \quad i = 1, \dots, n$
overdetermined if $n > 2$

(b) $\gamma\beta_1 R - b_1 P = \mu_1$
$\gamma\beta_2 R - b_2 P = \mu_2$

$$R = \frac{\begin{vmatrix} \mu_1 & -b_1 \\ \mu_2 & -b_2 \end{vmatrix}}{\gamma \begin{vmatrix} \beta_1 & -b_1 \\ \beta_2 & -b_2 \end{vmatrix}} = \frac{1}{\gamma} \frac{b_1\mu_2 - b_2\mu_1}{b_1\beta_2 - b_2\beta_1}$$

$$P = \frac{\gamma \begin{vmatrix} \beta_1 & \mu_1 \\ \beta_2 & \mu_2 \end{vmatrix}}{\gamma \begin{vmatrix} \beta_1 & -b_1 \\ \beta_2 & -b_2 \end{vmatrix}} = \frac{\beta_1\mu_2 - \beta_2\mu_1}{b_1\beta_2 - b_2\beta_1}$$

suppose $b_1\beta_2 - b_2\beta_1 > 0 \iff \frac{b_1}{\beta_1} > \frac{b_2}{\beta_2}$

$R > 0 \iff \frac{b_1}{\mu_1} > \frac{b_2}{\mu_2}$

$P > 0 \iff \frac{\beta_1}{\mu_1} > \frac{\beta_2}{\mu_2}$

or the opposite case $\rightarrow \frac{b_2}{\mu_2} > \frac{b_1}{\mu_1}$ & $\frac{\beta_2}{\mu_2} > \frac{\beta_1}{\mu_1}$

(i) $\beta_1 N_1 + \beta_2 N_2 = r(1 - \frac{R}{K})$
(iii) $b_1 N_1 + b_2 N_2 = \delta / c$

$$N_1 = \frac{\begin{vmatrix} r(1-\frac{R}{K}) & \beta_2 \\ \delta/c & b_2 \end{vmatrix}}{\begin{vmatrix} \beta_1 & \beta_2 \\ b_1 & b_2 \end{vmatrix}}, \quad N_2 = \frac{\begin{vmatrix} \beta_1 & r(1-\frac{R}{K}) \\ b_1 & \delta/c \end{vmatrix}}{\begin{vmatrix} \beta_1 & \beta_2 \\ b_1 & b_2 \end{vmatrix}}$$

suppose $\beta_1 b_2 - \beta_2 b_1 > 0$ "opposite case", then $N_i > 0$ iff
$\iff \frac{\beta_1}{b_1} > \frac{\beta_2}{b_2}$

$\frac{\beta_1}{b_1} \delta/c > r(1 - \frac{R}{K}) > \frac{\beta_2}{b_2} \delta/c$

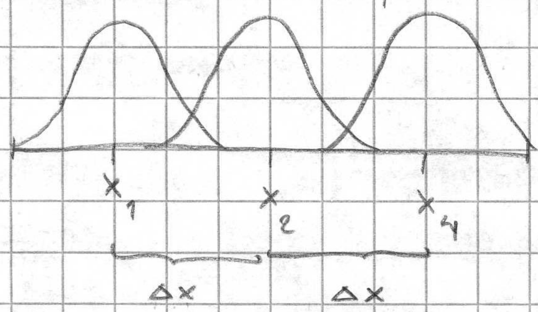
$\frac{\beta_1}{b_1} > \frac{c}{\delta} r(1 - \frac{R}{K}) > \frac{\beta_2}{b_2}$ $\quad$ c/δ can be chosen to satisfy this if $R < K \Leftarrow \gamma$ large enough

9.3. Limiting similarity of competing species

$$\frac{dN_i}{dt} = rN_i\left(1 - \frac{\sum_{j=1}^{n} \alpha(x_j - x_i)\hat{N}_j}{K}\right)$$

competition coefficient is: $\alpha(\Delta x) = \exp\left(-\frac{(\Delta x)^2}{2\sigma_\alpha^2}\right)$

We want to look at limiting similarity, i.e. minimum Δx such that three species can coexist within equal distances:



Let's find the condition for Δx such that the species 2 can invade the equilibrium of species 1 & 3. (This condition is enough, since it is competing most strongly).

Having 1 & 3 first at equilibrium, we have equations:

$$\hat{N}_1 + \alpha(2\Delta x)\hat{N}_3 = K$$
$$\hat{N}_3 + \alpha(2\Delta x)\hat{N}_1 = K$$

$$\Rightarrow \dots \Rightarrow \quad \hat{N}_1 = \hat{N}_2 = \frac{K}{1+\alpha(2\Delta x)}$$

Now species 2 can invade if:

$$1 - \frac{\alpha(\Delta x)\hat{N}_1 + \alpha(\Delta x)\hat{N}_3}{K} > 0$$

$$\Leftrightarrow \frac{\alpha(\Delta x)\hat{N}_1 + \alpha(\Delta x)\hat{N}_3}{K} < 1$$

use $\hat{N}_1 = \hat{N}_2$ & $K = \hat{N}_3 + \alpha(2\Delta x)\hat{N}_1$

$$\Leftrightarrow \frac{\alpha(\Delta x)\hat{N}_1 + \alpha(\Delta x)\hat{N}_1}{\hat{N}_1 + \alpha(2\Delta x)\hat{N}_1} < 1$$

$\alpha(\Delta x) = \exp\left(-\frac{(\Delta x)^2}{2\sigma^2}\right)$

$$\Leftrightarrow 2\alpha(\Delta x) < 1 + \alpha(2\Delta x)$$

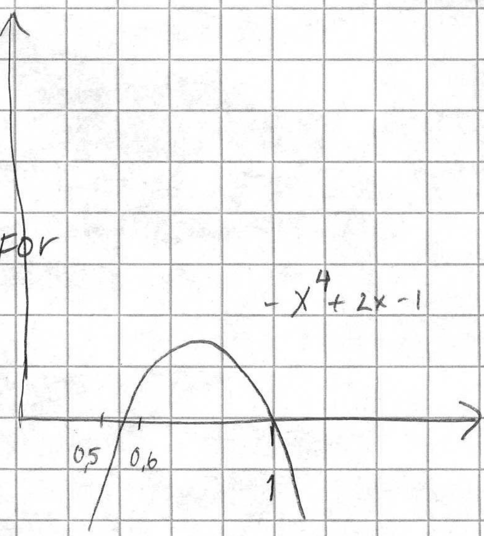
$\Rightarrow \alpha(2\Delta x) = \alpha(\Delta x)^4$

$$\Leftrightarrow -(\alpha(\Delta x))^4 + 2\alpha(\Delta x) - 1 < 0 \qquad (*)$$

9.3 continues.

Because $\alpha(\Delta x) = \exp\left(-\frac{(\Delta x)^2}{2\sigma^2}\right)$

we want the solutions of (*) for which $\alpha(\Delta x) < 1$.



Numerically one can see that smallest solution of (*) is between 0,5 and 0,6.

We say
$$\alpha(\Delta x) < 0.5$$
$$\exp\left(-\frac{(\Delta x)^2}{2\sigma^2}\right) < 0.5$$
$$\Rightarrow \quad -\frac{\Delta x^2}{2\sigma^2} < \ln 0.5$$
$$(\Delta x)^2 > -2\sigma^2 \ln 0.5$$
$$\underline{\underline{\Delta x > \sqrt{-2 \ln 0.5} \cdot \sigma}}$$