(8.1)

a.)

$$Q(N_1, N_2) = a_{11} a_{21} (N_1 - \hat{N}_1)^2 + 2 a_{12} a_{21} (N_1 - \hat{N}_1)(N_2 - \hat{N}_2) + a_{12} a_{22} (N_2 - \hat{N}_2)^2$$

- Q has clearly zero at $\hat{N}$.
- to prove the positivity, remember:

$$\boxed{\frac{a_{21}}{a_{11}} < \frac{\delta_2}{\delta_1} = k \quad \& \quad \frac{a_{22}}{a_{12}} > \frac{\delta_2}{\delta_1} = k}$$

look at kQ

$$kQ = k a_{11} a_{21} (\dots)^2 + 2k a_{21} a_{12} (\dots)(\dots) + k a_{12} a_{22} (\dots)^2$$

$$> a_{21}^2 (\dots) + 2k a_{21} a_{12} (\dots)(\dots) + k^2 a_{12}^2 (\dots) \qquad (*)$$

(using $a_{21} < a_{11} k$ and $\frac{a_{22}}{k} > a_{12}$)

and $(*) = \left(a_{21} (N_1 - \hat{N}_1) + k a_{12} (N_1 - \hat{N}_2) \right)^2 > 0$

For all $(N_1, N_2) \in \mathbb{R}^+_{int}$ Also, $Q(N_1, N_2)$ has zero at $\hat{N}$,

- $\dot{Q} = \left(2 a_{11} a_{21} (N_1 - \hat{N}_1) + 2 a_{12} a_{21} (N_2 - \hat{N}_2) \right) \cdot \left(\delta_1 - a_{11} N_1 - a_{12} N_2 \right)$
  $+ \dots$ (similar term for N_2)

$\delta = a_{11} \hat{N}_1 + a_{12} \hat{N}_2$ at equil.

$$= \left(2 a_{11} a_{21} (N_1 - \hat{N}_1) + 2 a_{12} a_{21} (N_2 - \hat{N}_2) \right) \left(a_{11} \hat{N}_1 + a_{12} \hat{N}_2 - a_{11} N_1 - a_{12} N_2 \right)$$

$$= 2 a_{21} \left(a_{11} (N_1 - \hat{N}_1) + a_{12} (N_2 - \hat{N}_2) \right) \left(a_{11} (\hat{N}_1 - N_1) + a_{12} (\hat{N}_2 - N_2) \right)$$

$$= \underbrace{-2 a_{21} \left(a_{11} (N_1 - \hat{N}_1) + a_{12} (N_2 - \hat{N}_2) \right)^2}_{< 0}$$

8.2. We have a system:

$$\begin{cases} \frac{dN}{dt} = rN\left(1-\frac{N}{k}\right) - \beta NP \\ \frac{dP}{dt} = \gamma\beta NP - \delta P \end{cases}$$

Multiplied with a Dulac-punction $\frac{1}{NP}$ it becomes:

$$\begin{cases} \frac{1}{NP}\frac{dN}{dt} = \frac{r}{P}\left(1-\frac{N}{K}\right) - \beta \\ \frac{1}{NP}\frac{dP}{dt} = \gamma\beta - \frac{\delta}{N}, \end{cases}$$

divergence of which is:

$$\mathrm{div}\left(\frac{1}{NP}\frac{dN}{dt}, \frac{1}{NP}\frac{dP}{dt}\right) = -\frac{r}{PK} < 0 \quad \text{whenever } P > 0$$

$\Rightarrow$ original system does not have a limit cycle.

(8.3) $$\frac{dN_i}{dt} = \left(\rho_i - \sum_{j=1}^{m} a_{ij} N_j\right) N_i \qquad i = 1, \dots, m$$

competitive coefficients are given by: $\qquad m$ consumers

$$(*) \qquad a_{ij} = \sum_{k=1}^{n} \gamma_{ik} \beta_{ik} \beta_{jk} L_k / S_k$$, where n is the number of resources

At Equilibrium, densities of competitors are given by:

$AN = \rho$, where a_{ij} is defined by $(*)$

a_{ij}'s can be expressed as:

$$A = \underbrace{\underbrace{\begin{pmatrix} \text{---}\ \frac{\beta_{jk} L_k}{S_k}\ \text{---} \end{pmatrix}}_{m \times n}}_{=:\,*} \underbrace{\begin{pmatrix} | \\ \gamma_{ik}\beta_{ik} \\ | \end{pmatrix}}_{n \times m} \qquad m > n$$

j'th species

i'th species

Now matrix $(*)$ has only m rows and since

$AN = \rho$ generically has no solution.

8.5. model:

$$\begin{cases} \frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) - \beta_1 N P_1 - \frac{\beta_2 N}{1+\beta_2 T N} P_2 \\ \frac{dP_1}{dt} = \gamma \beta_1 N P_1 - \delta P_1 \\ \frac{dP_2}{dt} = \frac{\gamma \beta_2 N}{1+\beta_2 T N} - \delta P_2 \end{cases}$$

a.) stability of the system with only consumer 1.

equilibrium: $\hat{N} = \frac{\delta}{\gamma \beta_1}$ & $\hat{P} = \frac{r\left(1-\frac{\delta}{\gamma\beta_1 K}\right)}{K}$

Jacobian
$$\begin{bmatrix} r - 2r\frac{\hat{N}}{K} - \beta\hat{P} & -\beta\hat{N} \\ \gamma\beta_1\hat{P} & 0 \end{bmatrix}$$

$\det(J) > 0$

$$\text{trace}(J) = r - 2r\left(\frac{\delta}{\gamma\beta_1 K}\right) - r + r\frac{\delta}{\gamma\beta_1 K}$$

$$= \underbrace{\frac{\delta}{\gamma\beta_1 K}\left(-2r + r\right)}_{<0} \Rightarrow \text{stable equilibria}$$

56.

$$\int_0^\tau \overbrace{\frac{\gamma \beta_2 N(t)}{1+\beta_2 T N(t)}}^{f(N(t))} dt = \delta \tau$$

avegage growth over time is the same that ~~on the limit cycle~~. the "average" death rate.

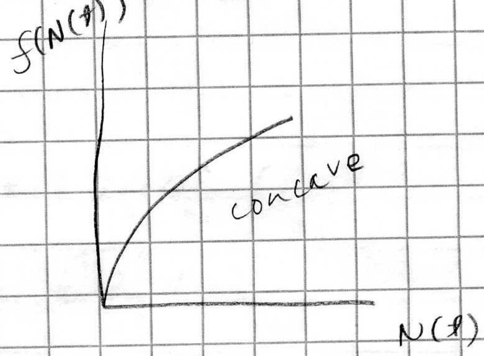
$$\odot \quad \frac{1}{\tau}\int_0^\tau \frac{d \ln P_2}{dt} d\tau = \frac{1}{\tau}\int_0^\tau f(N(t))\, dt - \delta$$

$$\langle f(N) \rangle = \delta$$

Jensen's inequality:

$$f(\hat{N}) = \langle f(N) \rangle < f(\bar{N})$$

$$\Rightarrow \hat{N}_2 < \bar{N}_{2\Pi}$$



$f(\hat{N}) = \delta$

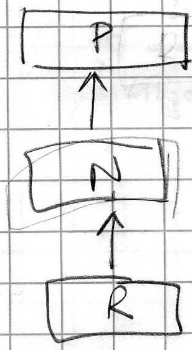
1. P_2 common $\Rightarrow$ $\hat{N}_2$, $\bar{N}$ $\Rightarrow$ $\hat{N}_2 < \bar{N}_2$

P_1 can grow $\hat{N}_1$ is needed $\Rightarrow$ $\hat{N}_1 < \bar{N}_2$

2. P_1 common $\hat{N}_1$

P_2 grows $\hat{N}_2 < \hat{N}_1$

8.4.



$$\begin{cases} \dfrac{dR}{dt} = rR\left(1-\dfrac{R}{K}\right) - c_1 N R \\[2ex] \dfrac{dN}{dt} = \gamma_1 c_1 N R - c_2 N P \\[2ex] \dfrac{dP}{dt} = \gamma_2 c_2 N P - \mu P \Rightarrow \hat{P} = \boxed{\dfrac{\gamma_2 c_2 \hat{N}}{\mu}} \end{cases}$$

At equilibrium $(\hat{R}, \hat{N}, \hat{P})$

$$\hat{R} = \frac{c_2 \hat{P}}{\gamma_1 c_1} \qquad \hat{P} = \frac{\gamma_2 c_2 \hat{N}}{\mu}$$

$$\hat{R} = \frac{\gamma_2 c_2^2}{\gamma_1 c_1 \mu} \hat{N}$$

finally $\hat{N} = \dfrac{r\left(1-\dfrac{\hat{R}}{K}\right)}{c_1}$

$$\hat{N} = \frac{r\left(1 - \dfrac{\gamma_2 c_2^2}{\gamma_1 c_1 \mu K}\hat{N}\right)}{c_1}$$

$$N = \frac{r}{c_1} - \frac{\gamma_2 c_2^2}{\gamma_1 c_1^2 \mu K}\hat{N}$$