7.1. SIS-model:

$$\begin{cases} \dfrac{dS}{dt} = -\beta S I + \gamma I \\ \dfrac{dI}{dt} = \beta S I - \gamma I \end{cases}$$

Observe: $\dfrac{dS}{dt} + \dfrac{dI}{dt} = \dfrac{d}{dt}(S+I) = 0$

$\Rightarrow$ Denote: $S + I = N$ (constant)

Thus:

$$\begin{aligned} \frac{dI}{dt} &= \beta(N-I)I - \gamma I \\ &= (\beta N - \gamma - \beta I) I \\ &= \beta\left(N - \frac{\gamma}{\beta} - I\right) I \\ &= \beta\left(N - \frac{\gamma}{\beta}\right)\left(1 - \frac{I}{N - \frac{\gamma}{\beta}}\right) I \end{aligned}$$

Which is the same form as logistic growth model

$$\frac{dN}{dt} = r\left(1 - \frac{N}{k}\right) N$$

The epidemiological
fitness cost of
dr

7.2.

$$\frac{dS}{dt} = B - \beta S I + \delta R - \mu S$$

$$\frac{dI}{dt} = \beta S I - \gamma I - (\mu + \alpha) I$$

$$\frac{dR}{dt} = \gamma I - \delta R - \mu R$$

R_0 := the expected number of infections one infective produces in a totally susceptible population.

For calculating R_0 we need

- expected time an individual is infected: $\frac{1}{\gamma+\mu+\alpha}$

- the rate at which it produces infections when all the others are susceptible, i.e. $S \approx N$. This is: βN_0

$$\Rightarrow R_0 = \beta N_0 \cdot \frac{1}{\gamma+\mu+\alpha} = \frac{\beta N_0}{\gamma+\mu+\alpha}$$

Disease is viable when $\frac{\beta N_0}{\gamma+\mu+\alpha} > 1$

Here N_0 is the equilibrium site of the population, in absence of disease:

$B - \mu N_0 = 0$

$$\boxed{N_0 = \frac{B}{\mu}}$$

For endemic equilibrium, we need to investigate the equilibrium of the system: ($\delta = R$)

$$\begin{cases} \frac{dS}{dt} = B - \beta S I - \mu S \\ \frac{dI}{dt} = \beta S I - \gamma I - (\mu+\alpha) I \\ \frac{dR}{dt} = \gamma I - \mu R \end{cases}$$

since R does not appear in equations for I and S, we only need to analyze the system for I and S.

$$\begin{cases} B - \beta S I - \mu S = 0 \\ \beta S I - \gamma I - (\mu + \alpha) I = 0 \end{cases}$$

$\hat{I}$ - isocline: $\hat{I}(\beta S - \gamma - \mu - \alpha) = 0$ $\quad \hat{I} = 0$

or $\beta S = \gamma + \mu + \alpha$

$$\hat{S} = \frac{\gamma + \mu + \alpha}{\beta}$$

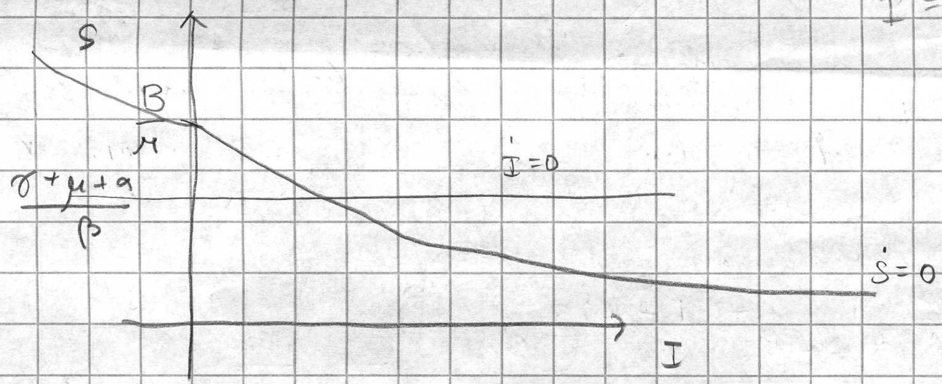
$\hat{S}$ - isocline: $S(-\beta I - \mu) = -B$

$$S = \frac{B}{\beta I + \mu} > 0 \quad \text{For all } I.$$

$$\beta I + \mu = \frac{B}{S}$$

$$I = \frac{\frac{B}{S} - \mu}{\beta}$$

In (I, S) -plane



For the intersection point to be in positive quadrant we need:

$$\frac{\gamma + \mu + \alpha}{\beta} < \underbrace{\frac{B}{\mu}}_{:= N_0}$$

$$\iff \quad 1 < \frac{N_0 \beta}{\gamma + \mu + \alpha}$$

which was exactly the condition for the disease to be viable.

Jacobian: at endemic eq. $J = \begin{pmatrix} -\beta\hat{I} - \mu & -\beta\hat{S} \\ \beta\hat{I} & 0 \end{pmatrix}$ $\quad$ trace < 0, det > 0

$\Rightarrow$ Stable equilib

$\boxed{7.3.}$ If we assume that disease is at its endemic steady state, we can consider force of infection $\Lambda := \beta \hat{I}$ to be constant

When survival is exponential we have

$$\int_0^\infty N(a)\, da = \frac{N_0}{\mu}, \text{ where } a \text{ is age}$$

$$\int_0^\infty S(a)\, da = \frac{N_0}{\Lambda + \mu}$$

At endemic steady state:

$$R_0 \cdot \frac{\hat{S}}{\hat{N}} = 1$$

$$R_0 \cdot \frac{\mu}{\Lambda + \mu} = 1 \iff \Lambda = R_0 \mu - \mu$$

$$\Lambda = \mu (R_0 - 1)$$

This is the force of infection, i.e the average rate at which individuals get infected. So the inverse of that is the average age at infection

$$\bar{a} = \frac{1}{\mu(R_0 - 1)} = \frac{L}{(R_0 - 1)}$$

What vaccination does to $\bar{a}$? Remember that $\Lambda = \beta \hat{I}$. We can solve from equations that

$$\underbrace{\hat{S} = \frac{\gamma + \mu}{\beta}}_{\text{no effect}} \quad \text{and} \quad \underbrace{\hat{I} = \frac{B}{\gamma + \mu} - \frac{\mu}{\gamma + \mu}\hat{S}}_{}$$

vaccination decreases $B \Rightarrow \hat{I}$ decreases

$\Rightarrow \Lambda$ decreases.

$\Rightarrow a = \frac{1}{\Lambda}$ increases.

$$\begin{cases} \dfrac{dN}{dt} = bN^2\left(1-\dfrac{N}{M}\right) - \mu N - \alpha I \\ \dfrac{dI}{dt} = \beta(N-I)I - (\alpha+\mu)I \end{cases}$$

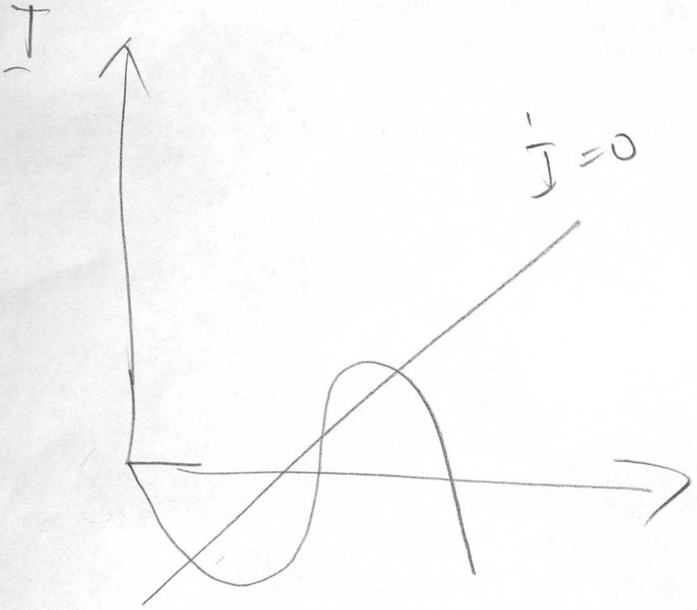
$$\dot{I}=0 \iff I = N - \frac{\alpha+\mu}{\beta}$$

$$\dot{N}=0 \iff I = \Big(\underbrace{\frac{b}{\alpha}N\left(1-\frac{N}{M}\right)-\frac{\mu}{\alpha}}_{\text{zero when}}\Big)N$$

We could have:

$$N = -b \pm \sqrt{b^2 - 4\frac{b}{M}\cdot\mu}$$

$$\iff b \geq 4\frac{\mu}{M}$$



or