11.1. Assume we have two strategies, $\overset{1}{\cancel{A}}$ & $\overset{2}{\cancel{B}}$.
Denote x_1 = frequency of A $\Rightarrow$ $1-x_1$ is the frequency of B.

· Fitnesses for $\overset{1}{\cancel{A}}$ & $\overset{2}{\cancel{B}}$

Payoff matrix: $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$F_1 = x_1 \cdot a_{11} + (1-x_1) a_{12}$$

$$F_2 = x_1 \cdot a_{21} + (1-x_1) a_{22}$$

$\Rightarrow$ average fitness: $x_1 \cdot F_1 + (1-x_1) F_2 = \bar{F}$

Replicator dynamics is:

$$\frac{dx_1}{dt} = x_1 \left[F_1 - \bar{F} \right] = x_1 \left[F_1 - x_1 F_1 - (1-x_1) F_2 \right]$$

$$= x_1 (1-x_1) \left[F_1 - F_2 \right]$$

$$= x_1 (1-x_1) \left[x_1 (a_{11} - a_{21}) + (1-x_1)(a_{12} - a_{22}) \right]$$

"$\Leftarrow$" if 1 is an ESS then $a_{11} > a_{21}$ and we can choose $1-x$ so that $\frac{dx_1}{dt} > 0$ $\Rightarrow$ 1 is stable.

"$\Rightarrow$" If $\hat{x}_1 = 1$ is stable then we have a such $\varepsilon = (1-x_1)$

$\frac{dx_1}{dt} > 0$, which shows $x_1 = 1$ is an ESS.

11.2. In this exercise the purpose is to analyse all possible solutions of equations:

$$\begin{cases} 5\hat{p}_2 - 4\hat{p}_3 = pAp \\ -7\hat{p}_1 + 8\hat{p}_3 = pAp \\ -\hat{p}_1 + 2\hat{p}_2 = pAp \end{cases}$$

and check whether they could be ESS's from ESS conditions.
Especially, one can find a mixed solution of equation above with $\hat{p}_1 = \hat{p}_2 = \hat{p}_3 = \frac{1}{3}$, but show that this is not an ESS.

11.3. - One can easily see that H/H and D/D cannot be pure ESS's.
- On the other hand, H/D and D/H are pure ESS's
- with ESS conditions, show that a mixed strategy of H/H and D/D cannot be stable.

11.4. a) $$\frac{dN_i}{dt} = \left(\rho - \sum_{j=1}^{n} a_{ij} N_j\right) N_i \qquad i = 1, \dots, n$$

define $N(t) = \sum_{i=1}^{n} N_i(t)$ & $x_i(t) = \frac{N_i(t)}{N(t)}$

$$\frac{dN_i}{dt} = \left(\rho - \sum_{j=1}^{n} a_{ij} N_j\right) N_i$$

$$\Leftrightarrow \quad \frac{dx_i}{dt} = \left(\rho - \sum_{j=1}^{n} a_{ij} N_j\right) x_i$$

$$\Leftrightarrow \quad \frac{dx_i}{dt} = \left(\frac{\rho}{N} - \underbrace{\sum_{j=1}^{n} a_{ij} x_j}_{=[Ax]_i}\right) N x_i$$

We can show that $\frac{\rho}{N} = xAx$:

$$\underbrace{\sum_{i=1}^{n} \frac{dx_i}{dt}}_{=0} = \sum_{i=1}^{n} \frac{\rho}{N} \cdot N x_i - N \sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ij} x_j$$

$$\Leftrightarrow \quad 0 = \rho - N \underbrace{\sum_{i=1}^{n} x_i \sum_{j=1}^{n} a_{ij} x_j}_{= xAx}$$

$$\Rightarrow \quad xAx = \frac{\rho}{N}.$$

$\Rightarrow$ original system can be written as:

$$\frac{dx_i}{dt} = \left(xAx - [Ax]_i\right) N x_i$$

and changing the sign of matrix A one can obtain replicator equation.

b.) Show that any n-dimensional Lotka-Volterra-model can be rewritten as n+1 dimensional Lotka-Volterra ~~model~~ system with equal intrinsic growth rates.

$$\frac{dx_i}{dt} = \left(\rho_i - \sum a_{ij} N_j\right) x_i$$

$$\frac{dx_i}{dt} = \left(\underbrace{\bar{\rho}} - \sum a_{ij} N_j - \underbrace{q_i \cdot 1}\right) x_i$$

add to each equation term q_i to make $\boxed{\bar{\rho} = \rho_i + q_i}$ for every equation. Then substract q_i from each equation

now for every i the intrinsic growth rate is the same.

there appears a competitor with frequency one, and comp. coefficient q_i

Only thing we need now is to add the competitor with frequency 1 to the system.

$$\frac{dx_{n+1}}{dt} = \left(\underbrace{\bar{\rho} - \bar{\rho} N_{n+1}}\right) x_{n+1}$$

has equilibrium at $\bar{N} = 1$ and does not compete with others.

Further, it also has the same intrinsic growth rate.