10.1. Assume $D(x^*) = 0$ and $D(x) = \frac{\partial}{\partial y} S_x(y) \Big|_{y=x}$

a.) Show that x^* is evolutionary stable if:
$$\frac{\partial^2}{\partial y^2} S_x(y) \Big|_{y=x=x^*} < 0.$$

A singular strategy is evol. stable if no initially rare mutant can invade
$\Leftrightarrow$ $S_{x^*}(y) < 0$ for all $y \neq x^*$.
Since $S_{x^*}(x^*) = 0$, it is a maximum.

$$\Rightarrow \frac{\partial S_x(y)}{\partial y^2} < 0. \qquad (*)$$

b.) For a singular to be convergence stable, the selection gradient must point towards the singular strategy $\rightarrow$ it changes sign from positive to negative when going through x^*.

$$\text{So } D(x) = \frac{\partial S_x(y)}{\partial y} \Big|_{y=x} < 0$$

$$\frac{dD(x)}{dx} = \frac{\partial^2 S}{\partial x \partial y} \Big|_{y=x} + \frac{\partial^2 S}{\partial y^2} \Big|_{y=x} < 0$$

$$\Rightarrow \frac{dD(x)}{dx} \Big|_{x=x^*} = \frac{\partial^2 S}{\partial x \partial y} \Big|_{y=x=x^*} + \frac{\partial^2 S}{\partial y^2} \Big|_{y=x=x^*} < 0 \quad (☺)$$

c.) For evolutionary branching we need convergence stability of singularity, but not evolutionary stability

$\Rightarrow$ (☺) should hold
$\Rightarrow$ $(*)$ should NOT hold

$$\Rightarrow \frac{\partial^2 S_x(y)}{\partial x \partial y} < 0.$$

10.2.

- exponential growth rate of mutant:

$$\rho(x_m) - \alpha(x_m - x_r)\hat{N}_r$$

- resident's equilibrium density:

$$\hat{N}_r = \frac{-ax_r + b}{1 - \frac{1}{1+\nu}}$$

- For x^* to be singular, we need $D(x^*) = 0$.
- Exponential growth rate of mutant differentiated w.r.t. x_m:

$$(1.) \quad -a + (-k)\cdot(-1)\cdot \frac{\nu \exp(\ldots)}{(1+\nu\exp(\ldots))^2} \cdot \frac{(-ax_r + b)}{1 - \frac{1}{1+\nu}} \quad \Bigg| \text{ evaluate at: } x_r = x_m = x^*$$

$$= \ldots = -a \frac{\nu k(-ax^* + b)}{\nu(1+\nu)}$$

What's above is linear for x^* $\rightarrow$ a unique solution for it being zero exists.

Differentiating (1.) further w.r.t. x_m:

$$\frac{\partial^2}{\partial x_m^2} S_{x_r}(x_m) = \ldots \propto \frac{k^2(-\nu + \nu^2)}{(1 + \nu\exp)^3}$$

positive when $\nu > 1$, and negative before that.

b.) Construct a decreasing function ρ such that there is a singular trait value that is not convergence stable.

Assume we have a growth rate function g at singularity, we need.

$$g'(x^*) = \frac{-\nu k}{\nu(1+\nu)} g(x^*)$$

Also, we need conditions from ex. 10.1. b to be satisfied in this singularity.

10.3.

	Coop	Defect
Coop	$B-C$	$-C$
Defect	B	0

$B > B-C \Rightarrow$ Coop is not an ESS

$0 > -C \Rightarrow$ Defect is an ESS

Can there be a mixed strategy $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = p$? For such we would have:

$$\begin{cases} (Ap)_1 = (B-C)p_1 - C p_2 = D \\ (Ap)_2 = B p_1 = D \end{cases}$$ (some constant)

But for such system no positive solution exists

$\Rightarrow$ Defect is the only ESS!

10.4.

2 × 2 matrix games

	R_1	R_2
R_1	a	b
R_2	c	d

- When is R_1 an ESS? When $a > c$
- R_2 is ESS when $d > b$
- mixed ESS? $\begin{pmatrix} p \\ 1-p \end{pmatrix} = \underline{p}$

$$(A\underline{p})_1 = (A\underline{p})_2 \Rightarrow pa + (1-p)b = pc + (1-p)d$$

$$\Rightarrow p = \frac{d-b}{a-b-c+d} \longleftarrow \text{a candidate!}$$

- Is $p \in]0,1[$?

- Is the other ESS condition satisfied?

$pAq > qAq$ $\quad$ q is a strategy other than p.

$$\Leftrightarrow pqa + p(1-q)b + (1-p)qc + (1-p)(1-q)d > q^2a + q(1-q)b + (1-q)qc + (1-q)^2d$$

$$\Leftrightarrow (p-q)\left[q(a-b-c+d) + b-d\right] > 0 \qquad \Big|\ p = \frac{d-b}{a-b-c+d}$$

$$(p-q)\left[(a-b-c+d)(q-p)\right]$$

$$= \underline{-(q-p)^2\left[a-b-c+d\right]} \longrightarrow \underline{\text{This has to be negative}}$$

$\Rightarrow$ For $p \in [0,1]$ $\quad d-b<0$

$\Rightarrow a < c$

10.5. Find all ESS of game

	R_1	R_2	R_3
R_1	0	5	-4
R_2	-7	0	8
R_3	-1	2	0

pure ESS's: only R_1 is a pure ESS.

Subcorollary of Bishop-Cannings-theorem states that if p^* is ESS → if q is such that $\text{supp}(q) \subseteq \text{supp}(p^*)$ → q is not an ESS.

⇒ ~~[illegible]~~ No full strategy can be ESS or a pair of strategies that contain R_1 can be ESS.

⇒ Only mixed strategy $\{R_2, R_3\}$ can be ESS.

→ Let $\underline{p} = \begin{pmatrix} p \\ 1-p \end{pmatrix}$

$$(1-p)\,8 = p\,2$$
$$8 = 10p$$
$$p = \frac{8}{10} \Rightarrow 1-p = \frac{2}{10}.$$

→ Is $p A q > q A q$ for all $q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$

$$p \cdot \begin{pmatrix} 5q_2 - 4q_3 \\ -7q_1 + 8q_3 \\ -1q_1 + 2q_2 \end{pmatrix} > q_1(5q_2 - 4q_3) + q_2(-7q_1 + 8q_3) + q_3(-q_1 + 2q_2)$$

$$\frac{8}{10}(-7q_1 + 8q_3) + \frac{2}{10}(-q_1 + 2q_2) > \ldots$$

We see that for $q = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ this does not hold. → $\boxed{\text{not an ESS!}}$