2 Representations of finite groups

As the very first thing, we take a brief look at the classical topic of representations of finite groups. Many things are easier than later in the course when we discuss representations of "quantum groups". The most important result is that all finite dimensional representations are direct sums of irreducible representations, of which there are only finitely many.

Reminders about groups and related concepts

Definition 1. *A group is a pair $(G, \circ)$, where G is a set and $\circ$ is a binary operation on G*

$$\circ : G \times G \to G \qquad (g, h) \mapsto g \circ h$$

such that the following hold

"Associativity": $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$ for all $g_1, g_2, g_3 \in G$

"Neutral element": there exists an element $e \in G$ s.t. for all $g \in G$ we have $g \circ e = g = e \circ g$

"Inverse": for any $g \in G$, there exists an element $g^{-1} \in G$ such that $g \circ g^{-1} = e = g^{-1} \circ g$

A group $(G, \circ)$ is said to be finite if its order $|G|$ (that is the cardinality of G) is finite.

We usually omit the notation for the binary operation $\circ$ and write simply $gh := g \circ h$. For abelian groups we often use the additive symbol $+$.

Also, we usually abbreviate and write only G for the group $(G, \circ)$.

Example 1. *Let X be a set. Then $S(X) := \{\sigma : X \to X \text{ bijective}\}$ with composition of functions is a group, called the symmetric group of X.*

In the case $X = \{1, 2, 3, \ldots, n\}$ we denote the symmetric group by S_n.

Example 2. *Let V be a vector space and $\mathrm{GL}(V) = \mathrm{Aut}(V) = \{A : V \to V \text{ linear bijection}\}$ with composition of functions as the binary operation. Then $\mathrm{GL}(V)$ is a group, called the general linear group of V (or the automorphism group of V). When V is finite dimensional, $\dim(V) = n$, and a basis of V has been chosen, then $\mathrm{GL}(V)$ can be identified with the group of $n \times n$ matrices having nonzero determinant, with matrix product as the group operation.*

Let $\mathbb{K}$ be the ground field and $V = \mathbb{K}^n$ the standard n-dimensional vector space. In this case we denote $\mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{K})$.

Example 3. *The group D_4 of symmetries of a square, or the dihedral group of order 8, is the group with two generators*

$$r \text{ "rotation by } \pi/2\text{"} \qquad m \text{ "reflection"}$$

and relations

$$r^4 = e \qquad m^2 = e \qquad rmrm = e.$$

Definition 2. *Let $(G_1, \circ_1)$ and $(G_2, \circ_2)$ be groups. A mapping $f : G_1 \to G_2$ is said to be a (group) homomorphism if for all $g, h \in G_1$*

$$f(g \circ_1 h) = f(g) \circ_2 f(h).$$

Example 4. *The determinant function $A \mapsto \det(A)$ from the matrix group $\mathrm{GL}_n(\mathbb{C})$ to the multiplicative group $\mathbb{C}^*$ of non-zero complex numbers, is a homomorphism since $\det(A\,B) = \det(A)\,\det(B)$.*

We will assume that the participants are familiar with the notions of subgroup, normal subgroup, quotient group, canonical projection, kernel, isomorphism etc.

One of the most fundamental recurrent principles in mathematics is the isomorphism theorem. We recall that in the case of groups it states the following.

Theorem 1. *Let G and H be groups and $f : G \to H$ a homomorphism. Then*

1°) $\mathrm{Im}\,(f) := f(G) \subset H$ *is a subgroup.*

2°) $\mathrm{Ker}\,(f) := f^{-1}(\{e_H\}) \subset G$ *is a normal subgroup.*

3°) The quotient group $G/\mathrm{Ker}\,(f)$ *is isomorphic to* $\mathrm{Im}\,(f)$.

More precisely, there exists an injective homomorphism $\bar{f} : G/\mathrm{Ker}\,(f) \to \mathrm{Im}\,(f)$ *such that the following diagram commutes*

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "\pi"'] & & H \\ & G/\mathrm{Ker}\,(f) \arrow[ur, dashed, "\bar{f}"'] & \end{tikzcd},$$

where $\pi : G \to G/\mathrm{Ker}\,(f)$ *is the canonical projection.*

The reader has surely encountered isomorphism theorems for several algebraic structures already — the following table summarizes the corresponding concepts in a few familiar cases

Structure	**Morphism** f	**Image** $\mathrm{Im}\,(f)$	**Kernel** $\mathrm{Ker}\,(f)$
group	group homomorphism	subgroup	normal subgroup
vector space	linear map	vector subspace	vector subspace
ring	ring homomorphism	subring	ideal
$\vdots$	$\vdots$	$\vdots$	$\vdots$

We will encounter isomorphism theorems for yet many other algebraic structures during this course: representations (modules), algebras, coalgebras, bialgebras, Hopf algebras, ... The idea is always the same, and the proofs only vary slightly, so we will probably not give full details in all cases.

A word of warning: since kernels, images, quotients etc. of different algebraic structures are philosophically so similar, we use the same notation for all, and assume that a reader sees that $\mathrm{Ker}\,(\rho(g))$ usually means the kernel of a linear map $\rho(g)$ (a vector subspace), whereas $\mathrm{Ker}\,(\rho)$ typically means the kernel of a group homomorphism ρ (a normal subgroup) and so on.

Representations: Definition and first examples

Definition 3. *Let G be a group and V a vector space. A representation of G in V is a group homomorphism* $G \to \mathrm{GL}(V)$.

Suppose $\rho : G \to \mathrm{GL}(V)$ is a representation. For any $g \in G$, the image $\rho(g)$ is a linear map $V \to V$. When the representation ρ is clear from context (and maybe also when it is not), we denote the images of vectors by this linear map simply by $g.v := \rho(g)\, v \in V$, for $v \in V$. With this notation the requirement that ρ is a homomorphism reads $(g\,h).v = g.(h.v)$. It is convenient to interpret this as a left multiplication of vectors $v \in V$ by elements g of the group G. Thus interpreted, we say that V is a (left) G-module.

Example 5. *Let V be a vector space and set $\rho(g) = \mathrm{id}_V$ for all $g \in G$. This is called the trivial representation of G in V. If no other vector space is clear from the context, the trivial representation means the trivial representation in the one dimensional vector space $V = \mathbb{K}$.*

Example 6. *The symmetric group S_n for $n \geq 2$ has another one dimensional representation called the* <u>*alternating representation*</u>*. This is the representation given by $\rho(\sigma) = \mathrm{sign}(\sigma)\ \mathrm{id}_{\mathbb{K}}$, where $\mathrm{sign}(\sigma)$ is minus one when the permutation σ is the product of odd number of transpositions, and plus one when σ is the product of even number of transpositions.*

Example 7. *Let D_4 be the dihedral group of order 8, with generators r, m and relations $r^4 = e$, $m^2 = e$, $rmrm = e$. Define the matrices*

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $R^4 = \mathbb{I}$, $M^2 = \mathbb{I}$, $RMRM = \mathbb{I}$, there exists a homomorphism $\rho : D_4 \to \mathrm{GL}_2(\mathbb{R})$ such that $\rho(r) = R$, $\rho(m) = M$. Such a homomorphism is unique since we have given the values of it on generators r, m of D_4. If we think of the square in the plane $\mathbb{R}^2$ with vertices $A = (1, 0)$, $B = (0, 1)$, $C = (-1, 0)$, $D = (0, -1)$, then the linear maps $\rho(g)$, $g \in D_4$, are precisely the eight isometries of the plane which preserve the square $ABCD$. Thus it is very natural to represent the group D_4 in a two dimensional vector space!

A representation ρ is said to be <u>faithful</u> if it is injective, i.e. if $\mathrm{Ker}\ (\rho) = \{e\}$. The representation of the symmetry group of the square in the last example is faithful, it could be taken as a defining representation of D_4.

When the ground field is $\mathbb{C}$, we might want to write the linear maps $\rho(g) : V \to V$ in their Jordan canonical form. But we observe immediately that the situation is as good as it could get:

Lemma 2. *Let G be a finite group, V a finite dimensional (complex) vector space, and ρ a representation of G in V. Then, for any $g \in G$, the linear map $\rho(g) : V \to V$ is diagonalizable.*

Proof. Observe that $g^n = e$ for some positive integer n (for example the order of the element g or the order of the group G). Thus we have $\rho(g)^n = \rho(g^n) = \rho(e) = \mathrm{id}_V$. This says that the minimal polynomial of $\rho(g)$ divides $x^n - 1$, which only has roots of multiplicity one. Therefore the Jordan normal form of $\rho(g)$ can only have blocks of size one. □

We still continue with an example (or definition) of representation that will serve as useful tool later.

Example 8. *Let ρ_1, ρ_2 be two representations of a group G in vector spaces V_1, V_2, respectively. Then the space of linear maps between the two representations*

$$\mathrm{Hom}(V_1, V_2) = \{T : V_1 \to V_2 \text{ linear}\}$$

becomes a representation by setting

$$g.T \;=\; = \rho_2(g) \circ T \circ \rho_1(g^{-1})$$

for all $T \in \mathrm{Hom}(V_1, V_2)$, $g \in G$. As usual, we often omit the explicit notation for the representations ρ_1, ρ_2, and write simply

$$(g.T)(v) = g.\big(T(g^{-1}.v)\big) \qquad \text{for any } v \in V_1.$$

To check that this indeed defines a representation, we compute

$$\big(g_1.(g_2.T)\big)(v) = g_1.\big((g_2.T)(g_1^{-1}.v)\big) = g_1.g_2.\big(T(g_2^{-1}.g_1^{-1}.v)\big) = g_1g_2.\big(T\big((g_1g_2)^{-1}.v\big)\big) = \big((g_1g_2).T\big)(v).$$

Definition 4. *Let G be a group and V_1, V_2 two G-modules (=representations). A linear map $T : V_1 \to V_2$ is said to be a* <u>*G-module map*</u> *(sometimes also called a G-linear map) if $T(g.v) = g.T(v)$ for all $g \in G$, $v \in V$.*

Note that $T \in \mathrm{Hom}(V_1, V_2)$ is a G-module map if and only if $g.T = T$ for all $g \in G$, when we use the representation of Example 8 on $\mathrm{Hom}(V_1, V_2)$. We denote by $\mathrm{Hom}_G(V_1, V_2) \subset \mathrm{Hom}(V_1, V_2)$ the space of G-module maps from V_1 to V_2.

Subrepresentations, irreducibility and complete reducibility

Definition 5. *Let ρ be a representation of G in V. If $V' \subset V$ is a subspace and if $\rho(g)\, V' \subset V'$ for all $g \in G$ (we say that V' is an invariant subspace), then taking the restriction to the invariant subspace, $g \mapsto \rho(g)|_{V'}$ defines a representation of G in V' called a subrepresentation of ρ.*

We also call V' a submodule of the G-module V.

The subspaces $\{0\} \subset V$ and $V \subset V$ are always submodules.

Example 9. *Let $T : V_1 \to V_2$ be a G-module map. The image* $\mathrm{Im}\,(T) = T(V_1) \subset V_2$ *is a submodule, since a general vector of the image can be written as $w = T(v)$, and $g.w = g.T(v) = T(g.v) \in$* $\mathrm{Im}\,(T)$*. The kernel* $\mathrm{Ker}\,(T) = T^{-1}(\{0\}) \subset V_1$ *is a submodule, too, since if $T(v) = 0$ then $T(g.v) = g.T(v) = g.0 = 0$.*

Example 10. *When we consider* $\mathrm{Hom}(V_1, V_2)$ *as a representation as in Example 8, the subspace* $\mathrm{Hom}_G(V_1, V_2) \subset \mathrm{Hom}(V_1, V_2)$ *of G-module maps is a subrepresentation, which, by the remark after Definition 4, is a trivial representation in the sense of Example 5.*

Definition 6. *Let $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$ be representations of G in vector spaces V_1 and V_2, respectively. Let $V = V_1 \oplus V_2$ be the direct sum vector space. The representation $\rho : G \to \mathrm{GL}(V)$ given by*

$$\rho(g)(v_1 + v_2) \;=\; \rho_1(g)v_1 + \rho_2(g)v_2 \qquad \text{when } v_1 \in V_1 \subset V,\ v_2 \in V_2 \subset V$$

is called the direct sum representation of ρ_1 and ρ_2.

Both V_1 and V_2 are submodules of $V_1 \oplus V_2$.

A key property of representations of finite groups is that any invariant subspace has a complementary invariant subspace in the following sense.

Proposition 3. *Let G be a finite group. If V' is a submodule of a G-module V, then there is a submodule $V'' \subset V$ such that $V = V' \oplus V''$ as a G-module.*

Proof. First choose any complementary vector subspace U for V', that is $U \subset V'$ such that $V = V' \oplus U$ as a vector space. Let $\pi' : V \to V'$ be the canonical projection corresponding to this direct sum, that is

$$\pi'(v' + u) \;=\; v' \qquad \text{when } v' \in V',\, u \in U.$$

Define

$$\pi(v) \;=\; \frac{1}{|G|} \sum_{g \in G} g.\pi'(g^{-1}.v).$$

Observe that $\pi|_{V'} = \mathrm{id}_{V'}$ and $\mathrm{Im}\,(\pi) \subset V'$, that is π is a projection from V to V'. If we set $V'' = \mathrm{Ker}\,(\pi)$, then at least $V = V' \oplus V''$ as a vector space. To show that V'' is a subrepresentation, it suffices to show that π is a G-module map. This is checked by doing the change of summation variable $\tilde{g} = h^{-1}g$ in the following

$$\pi(h.v) \;=\; \frac{1}{|G|} \sum_{g \in G} g.\pi'(g^{-1}.h.v) \;=\; \frac{1}{|G|} \sum_{g \in G} g.\pi'\Big((h^{-1}g)^{-1}.v\Big) \;=\; \frac{1}{|G|} \sum_{\tilde{g} \in G} h\tilde{g}.\pi'\Big(\tilde{g}^{-1}.v\Big) \;=\; h.\pi(v).$$

We conclude that $V'' = \mathrm{Ker}\,(\pi) \subset V$ is a subrepresentation and thus $V = V' \oplus V''$ as a representation. □

Definition 7. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation. If there are no other subrepresentations but those corresponding to $\{0\}$ and V, then we say that ρ is an irreducible representation, or that V is a simple G-module.*

Proposition 3, with an induction on dimension of the G-module V, gives the fundamental result about representations of finite groups called complete reducibility, as stated in the following.

Corollary 4. *Let G be a finite group and V a finite dimensional G-module. Then, as representations, we have*

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n,$$

where each subrepresentation $V_j \subset V$, $j = 1, 2, \ldots, n$, is an irreducible representation of G.

We also mention the basic result which says that there is not much freedom in constructing G-module maps between irreducible representations.

Lemma 5 (Schur's Lemma)**.** *If V and W are irreducible representations of a group G, and $T : V \to W$ is a G-module map, then*

(i) either $T = 0$ or T is an isomorphism

(ii) if $V = W$, then $T = \lambda \, \mathrm{id}_V$ for some $\lambda \in \mathbb{C}$.

Proof. If $\mathrm{Ker}\,(T) \neq \{0\}$, then by irreducibility of V we have $\mathrm{Ker}\,(T) = V$ and therefore $T = 0$. If $\mathrm{Ker}\,(T) = \{0\}$, then T is injective and by irreducibility of W we have $\mathrm{Im}\,(T) = W$, so T is also surjective. This proves *(i)*. To prove *(ii)*, pick any eigenvalue λ of T (here we need the ground field to be algebraically complete, for example the field $\mathbb{C}$ of complex numbers). Now consider the G-module map $T - \lambda \mathrm{id}_V$, which has a nontrivial kernel. The kernel must be the whole space by irreducibility, so $T - \lambda \, \mathrm{id}_V = 0$. □

Characters

In the rest of this section G is a finite group of order $|G|$ and all representations are assumed to be finite dimensional.

We have already seen the fundamental result of complete reducibility: any representation of G is a direct sum of irreducible representations. It might nevertheless not be clear yet how to concretely work with the representations. We now introduce a very powerful tool for the representation theory of finite groups: the character theory.

Definition 8. *For $\rho : G \to \mathrm{GL}(V)$ a representation, the character of the representation is the function $\chi_V : G \to \mathbb{C}$ given by*

$$\chi_V(g) = \mathrm{Tr}(\rho(g)).$$

Observe that we have

$$\chi_V(e) = \dim\,(V)$$

and for two group elements that are conjugates, $g_2 = h g_1 h^{-1}$, we have

$$\chi_V(g_2) = \mathrm{Tr}(\rho(g_2)) = \mathrm{Tr}\Big(\rho(h)\,\rho(g_1)\,\rho(h)^{-1}\Big) = \mathrm{Tr}(\rho(g_1)) = \chi_V(g_1).$$

Thus the value of a character is constant on each conjugacy class of G (such functions $G \to \mathbb{C}$ are called class functions).

Example 11. *We have seen three (irreducible) representations of the group S_3: the trivial representation U and the alternating representation U', both one dimensional, and the two-dimensional representation V in* Problem sheet 1: Exercise 3*. The conjugacy classes of symmetric groups are given by the cycle decompositions of a permutation, in particular for S_3 the conjugacy classes are*

$$\begin{aligned} \textit{identity} &: \{e\} \\ \textit{transpositions} &: \{(12), (13), (23)\} \\ \textit{3-cycles} &: \{(123), (132)\}\,. \end{aligned}$$

We can explicitly compute the trace of for example the transposition (12) *and the three cycle* (123) *to get the characters of these representations*

$$\begin{array}{c|ccc} & \chi(e) & \chi((12)) & \chi((123)) \\ \hline U & 1 & 1 & 1 \\ U' & 1 & -1 & 1 \\ V & 2 & 0 & -1 \end{array}.$$

Recall that we have seen how to make the dual V^* a representation (cf. *Problem sheet 1: Exercise* 2), and how to make direct sum $V_1 \oplus V_2$ a representation. We can also build representations by taking tensor products of representations.

Definition 9. *Let $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$ be two representations of G. We make the tensor product space $V_1 \otimes V_2$ a representation by setting for simple tensors*

$$\rho(g)\,(v_1 \otimes v_2) \;=\; (\rho_1(g)v_1) \otimes (\rho_2(g)v_2)$$

and extending the definition linearly to the whole of $V_1 \otimes V_2$. Clearly for simple tensors we have

$$\rho(h)\rho(g)\,(v_1 \otimes v_2) \;=\; \Big(\rho_1(h)\rho_1(g)v_1\Big) \otimes \Big(\rho_2(h)\rho_2(g)v_2\Big) \;=\; \Big(\rho_1(hg)v_1\Big) \otimes \Big(\rho_2(hg)v_2\Big) \;=\; \rho(hg)\,(v_1 \otimes v_2)$$

and since both sides are linear, we have $\rho(h)\rho(g)\,t = \rho(hg)\,t$ for all $t \in V_1 \otimes V_2$, so that $\rho : G \to \mathrm{GL}(V_1 \otimes V_2)$ is indeed a representation.

Let us now see how these operations affect characters.

Proposition 6. *Let V, V_1, V_2 be representations of G. Then we have*

(i) $\chi_{V^*}(g) \;=\; \overline{\chi_V(g)}$

(ii) $\chi_{V_1 \oplus V_2}(g) \;=\; \chi_{V_1}(g) + \chi_{V_2}(g)$

(iii) $\chi_{V_1 \otimes V_2}(g) \;=\; \chi_{V_1}(g)\,\chi_{V_2}(g)\,.$

Proof. Part (i) was done in *Problem sheet 1: Exercise 2*. For the other two, recall first that if $\rho : G \to \mathrm{GL}(V)$ is a representation, then $\rho(g)$ is diagonalizable by Lemma 2. Therefore there are $n = \dim\,(V)$ linearly independent eigenvectors with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, and the trace is the sum of these $\chi_V(g) = \sum_{j=1}^{n} \lambda_j$. Consider the representations $\rho_1 : G \to \mathrm{GL}(V_1)$, $\rho_2 : G \to \mathrm{GL}(V_2)$. For $g \in G$, take bases of eigenvectors of $\rho_1(g)$ and $\rho_2(g)$ for V_1 and V_2, respectively: if $n_1 = \dim\,(V_1)$ and $n_2 = \dim\,(V_2)$ let $v^{(1)}_\alpha$, $\alpha = 1, 2, \ldots, n_1$, be eigenvectors of $\rho_1(g)$ with eigenvalues $\lambda^{(1)}_\alpha$, and $v^{(2)}_\beta$, $\beta = 1, 2, \ldots, n_2$, eigenvectors of $\rho_2(g)$ with eigenvalues $\lambda^{(2)}_\beta$. To prove (ii) it suffices to note that $v^{(1)}_\alpha \in V_1 \subset V_1 \oplus V_2$ and $v^{(2)}_\alpha \in V_2 \subset V_1 \oplus V_2$ are the $n_1 + n_2 = \dim\,(V_1 \oplus V_2)$ linearly independent eigenvectors for the direct sum representation, and the eigenvalues are $\lambda^{(1)}_\alpha$ and $\lambda^{(2)}_\beta$. To prove (iii) note that the vectors $v^{(1)}_\alpha \otimes v^{(2)}_\beta$ are the $n_1 n_2 = \dim\,(V_1 \otimes V_2)$ linearly independent eigenvectors of $V_1 \otimes V_2$, and the eigenvalues are the products $\lambda^{(1)}_\alpha \lambda^{(2)}_\beta$, since

$$g.(v^{(1)}_\alpha \otimes v^{(2)}_\beta) \;=\; \Big(\rho_1(g).v^{(1)}_\alpha\Big) \otimes \Big(\rho_2(g).v^{(2)}_\beta\Big) \;=\; (\lambda^{(1)}_\alpha\, v^{(1)}_\alpha) \otimes (\lambda^{(2)}_\beta\, v^{(2)}_\beta) \;=\; \lambda^{(1)}_\alpha \lambda^{(2)}_\beta\, (v^{(1)}_\alpha \otimes v^{(2)}_\beta).$$

Therefore the character of the tensor product reads

$$\chi_{V_1 \otimes V_2}(g) \;=\; \sum_{\alpha,\beta} \lambda^{(1)}_\alpha\, \lambda^{(2)}_\beta \;=\; \Big(\sum_{\alpha=1}^{n_1} \lambda^{(1)}_\alpha\Big)\Big(\sum_{\beta=1}^{n_2} \lambda^{(2)}_\beta\Big) \;=\; \chi_{V_1}(g)\,\chi_{V_2}(g).$$

□

For V a representation of G, set

$$V^G = \left\{ v \in V \,\middle|\, g.v = v \;\forall g \in G \right\}.$$

Then $V^G \subset V$ is a subrepresentation, which is a trivial representation in the sense of Example 5. We define a linear map φ on V by

$$\varphi(v) = \frac{1}{|G|} \sum_{g \in G} g.v \qquad v \in V.$$

Proposition 7. *The map φ is a projection $V \to V^G$.*

Proof. Clearly if $v \in V^G$ then $\varphi(v) = v$, so we have $\varphi|_{V^G} = \mathrm{id}_{V^G}$. For any $h \in G$ and $v \in V$, use the change of variables $\tilde{g} = hg$ to compute

$$h.\varphi(v) = \frac{1}{|G|} \sum_{g \in G} hg.v = \frac{1}{|G|} \sum_{\tilde{g} \in G} \tilde{g}.v = \varphi(v),$$

so we have $\mathrm{Im}\,(\varphi) \subset V^G$. □

Thus we have an explicitly defined projection to the trivial part of any representation, and we have in particular

$$\dim\,(V^G) = \mathrm{Tr}(\varphi) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g).$$

Now suppose that V and W are two representations of G and consider the representation $\mathrm{Hom}(V, W)$. We have seen in *Problem sheet 2: Exercise 2* that $\mathrm{Hom}(V, W) \cong W \otimes V^*$ as a representation. In particular, we know how to compute the character

$$\chi_{\mathrm{Hom}(V,W)}(g) = \chi_{W \otimes V^*}(g) = \chi_W(g)\, \chi_{V^*}(g) = \overline{\chi_V(g)}\, \chi_W(g).$$

We've also seen that the trivial part of this representation consists of the G-module maps between V and W,

$$\mathrm{Hom}(V, W)^G = \mathrm{Hom}_G(V, W),$$

and we get the following almost innocent looking consequence

$$\dim\left(\mathrm{Hom}_G(V, W)\right) = \mathrm{Tr}(\varphi) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)}\, \chi_W(g).$$

Suppose now that V and W are irreducible. Then Schur's lemma says that when V and W are not isomorphic, there are no nonzero G-module maps $V \to W$, whereas the G-module maps from an irreducible representation to itself are scalar multiples of the identity, i.e.

$$\dim\left(\mathrm{Hom}_G(V, W)\right) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{otherwise} \end{cases}.$$

We have in fact obtained a very powerful result.

Theorem 8. *The following statements hold for irreducible representations of a finite group G.*

(i) If V and W are irreducible representations, then

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)}\, \chi_W(g) = \begin{cases} 1 & \textit{if } V \cong W \\ 0 & \textit{otherwise} \end{cases}.$$

(ii) Characters of (non-isomorphic) irreducible representations are linearly independent.

(iii) The number of (isomorphism classes of) irreducible representations is at most the number of conjugacy classes of G.

Proof. The statement (i) was proved above. We can interpret it as saying that the characters of irreducible representations are orthonormal with respect to the natural inner product $(\psi, \phi) = \frac{1}{|G|}\sum_{g \in G} \overline{\psi(g)}\, \phi(g)$ on the space $\mathbb{C}^G$ of $\mathbb{C}$-valued functions on G. The linear independence, (ii), follows at once. Since a character has constant value on each conjugacy class, an obvious upper bound on the number of linearly independent characters gives (iii). □

We proceed with further consequences.

Corollary 9. *Let W_α, $\alpha = 1, 2, \ldots, k$, be the distinct irreducible representations of G. Let V be any representation, and let m_α be the multiplicity of W_α when we use complete reducibility:*

$$V = \bigoplus_\alpha m_\alpha\, W_\alpha$$

Then we have

(i) The character χ_V determines V (up to isomorphism).

(ii) The multiplicities are given by

$$m_\alpha = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{W_\alpha}(g)}\, \chi_V(g).$$

(iii) We have

$$\frac{1}{|G|} \sum_{g \in G} |\chi_V(g)|^2 = \sum_\alpha m_\alpha^2.$$

(iv) The representation V is irreducible if and only if

$$\frac{1}{|G|} \sum_{g \in G} |\chi_V(g)|^2 = 1.$$

Proof. The character of V is by Proposition 6 given by $\chi_V(g) = \sum_\alpha m_\alpha\, \chi_{W_\alpha}(g)$. Now (ii) is obtained by taking the orthogonal projection to χ_{W_α}. In particular we obtain the (anticipated) fact that in complete reducibility the direct sum decomposition is unique up to permutation of the irreducible summands. We also see (i) immediately, and (iii) follows from the same formula combined with $\overline{\chi_V(g)}\, \chi_V(g) = |\chi_V(g)|^2$. Then (iv) is obvious in view of (iii). □

We get some more nice consequences when we consider the representation given in the following examples.

Example 12. *Consider the vector space $\mathbb{C}^G$ with basis $\{e_g \,|\, g \in G\}$. For any $g, h \in G$, set*

$$h.e_g = e_{hg}$$

and extend linearly. This defines a $|G|$-dimensional representation called the regular representation of G. We denote the regular representation here by $\mathbb{C}[G]$ because later we will put an algebra structure on this vector space to obtain the group algebra of G, and then this notation is standard.

Example 13. *More generally, following the same idea, if the group G acts on a set X, then we can define a representation on the vector space $\mathbb{C}^X$ with basis $\{e_x | x \in X\}$ by a linear extension of $g.e_x = e_{(g.x)}$. These kind of represetations are called permutation representations.*

It is obvious, when we write matrices in the basis $(e_x)_{x \in X}$ and compute traces, that $\chi_{\mathbb{C}^X}(g)$ is the number of elements $x \in X$ which are fixed by the action of g. In particular the character of the regular representation is

$$\chi_{\mathbb{C}[G]}(g) \; = \; \begin{cases} |G| & \text{if } g = e \\ 0 & \text{if } g \neq e \end{cases} .$$

We can then use Corollary 9 (ii) and compute, for any irreducible W_α,

$$\frac{1}{|G|} \sum_{g \in G} \overline{\chi_{W_\alpha}(g)} \, \chi_{\mathbb{C}[G]}(g) \; = \; \frac{1}{|G|} \overline{\chi_{W_\alpha}(e)} \, |G| \; = \; \dim \left(W_\alpha\right).$$

Thus any irreducible representation appears in the regular representation by multiplicity given by its dimension

$$\mathbb{C}[G] \; = \; \bigoplus_\alpha m_\alpha \, W_\alpha \qquad \text{where} \qquad m_\alpha = \dim \left(W_\alpha\right).$$

Considering in particular the dimensions of the two sides, and recalling $\dim \left(\mathbb{C}[G]\right) = |G|$, we get the following formula

$$\sum_\alpha \dim \left(W_\alpha\right)^2 \; = \; |G|.$$

Example 14. *The above formula can give useful and nontrivial information. Consider for example the group S_4, whose order is $|S_4| = 4! = 24$. We have seen the trivial and alternating representations of S_4, and since there are five conjugacy classes (identity, transposition, two disjoint transpositions, three-cycle, four-cycle), we know that there are at most three other irreducible representations S_4. From the above formula we see that the sum of squares of their dimensions is $|S_4| - 1^2 - 1^2 = 22$. Since 22 is not a square, there must remain more than one irreducible, and since 22 is also not a sum of two squares, there must in fact be three other irreducibles. The only way to write 22 as a sum of three squares is $22 = 2^2 + 3^2 + 3^2$, so we see that the three remaining irreducible representations have dimensions $2, 3, 3$.*