3 Algebras, coalgebras, bialgebras and Hopf algebras

Here we first define the algebraic structures to be studied in the rest of the course.

Algebras

By the standard definition, an algebra (which for us will mean an associative unital algebra) is a triple $(A, \circ, 1_A)$, where A is a vector space (over a field $\mathbb{K}$, usually $\mathbb{K} = \mathbb{C}$) and $\circ$ is a binary operation on A

$$\circ : A \times A \to A \qquad (a, b) \mapsto a \circ b$$

and 1_A is an element of A such that the following hold:

"Bilinearity": the map $\circ : A \times A \to A$ is bilinear

"Associativity": $a_1 \circ (a_2 \circ a_3) = (a_1 \circ a_2) \circ a_3$ for all $a_1, a_2, a_3 \in A$

"Unitality": for all $a \in A$ we have $a \circ 1_A = a = 1_A \circ a$

We usually omit the notation for the binary operation $\circ$ and write simply $ab := a \circ b$. The algebra is said to be commutative if $ab = ba$ for all $a, b \in A$.

We usually abbreviate and write only A for the algebra $(A, \circ, 1_A)$. An algebra $(A, \circ, 1_A)$ is said to be finite dimensional if the $\mathbb{K}$-vector space A is finite dimensional.

For an element $a \in A$, a left inverse of a is an element a' such that $a'\, a = 1_A$ and a right inverse of a is an element a'' such that $a\, a'' = 1_A$. An element is said to be invertible if it has both left and right inverses. In such a case the two have to be equal since $a'' = 1_A\, a'' = a'\, a\, a'' = a'\, 1_A = a'$, and we denote by a^{-1} the (left and right) inverse of a. These trivial properties will come in handy a bit later.

Similarly, the unit 1_A is uniquely determined by the unitality property.

Example 1. *Any field $\mathbb{K}$ is an algebra over itself (and moreover commutative).*

Example 2. *The algebra of polynomials (with coefficients in $\mathbb{K}$) in one indeterminate x is denoted by*

$$\mathbb{K}[x] \;:=\; \left\{ c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n \;\middle|\; n \in \mathbb{N},\; c_0, c_1, \ldots, c_n \in \mathbb{K} \right\}.$$

The product is the usual product of polynomials (commutative).

Example 3. *Let V be a vector space and $\mathrm{End}(V) = \mathrm{Hom}(V, V) = \{T : V \to V \text{ linear}\}$ the set of endomorphisms of V. Then $\mathrm{End}(V)$ is an algebra, with composition of functions as the binary operation, and the identity map id_V as the unit. When V is finite dimensional, $\dim (V) = n$, and a basis of V has been chosen, then $\mathrm{End}(V)$ can be identified with the algebra of $n \times n$ matrices, with matrix product as the binary operation.*

Example 4. *For G a group, let $\mathbb{K}[G]$ be the vector space with basis $(e_g)_{g \in G}$ and define the product by bilinearly extending*

$$e_g\, e_h \;=\; e_{gh}.$$

Then, $\mathbb{K}[G]$ is an algebra called the group algebra of G, the unit is e_e, where $e \in G$ is the neutral element of the group.

Definition 1. *Let $(A_1, \circ_1, 1_{A_1})$ and $(A_2, \circ_2, 1_{A_2})$ be algebras. A mapping $f : A_1 \to A_2$ is said to be a homomorphism of (unital) algebras if f is linear and $f(1_{A_1}) = 1_{A_2}$ and for all $a, b \in A_1$*

$$f(a \circ_1 b) \;=\; f(a) \circ_2 f(b).$$

Definition 2. *For A an algebra, a vector subspace $A' \subset A$ is called a* subalgebra *if $1_A \in A'$ and for all $a', b' \in A'$ we have $a'b' \in A'$. A vector subspace $J \subset A$ is called a* left ideal *(resp. right ideal, resp. two-sided ideal or simply* ideal*) if for all $a \in A$ and $k \in J$ we have $ak \in J$ (resp. $ka \in J$, resp. both).*

For J an ideal, the quotient vector space A/J becomes an algebra by setting

$$(a + J)(b + J) \; = \; ab + J$$

(which is well defined since the three last terms in $(a + k)(b + k') = ab + ak' + kb + kk'$, and are in the ideal if k and k' are).

The isomorphism theorem for algebras now states the following.

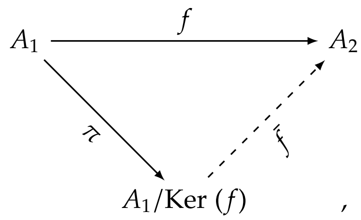
Theorem 1. *Let A_1 and A_2 be algebras and $f : A_1 \to A_2$ a homomorphism. Then*

1°) $\mathrm{Im}\,(f) := f(A_1) \subset A_2$ *is a subalgebra.*

2°) $\mathrm{Ker}\,(f) := f^{-1}(\{0\}) \subset A_1$ *is an ideal.*

3°) The quotient algebra $A_1/\mathrm{Ker}\,(f)$ *is isomorphic to* $\mathrm{Im}\,(f)$.

More precisely, there exists an injective algebra homomorphism $\bar{f} : A_1/\mathrm{Ker}\,(f) \to A_2$ such that the following diagram commutes



where $\pi : A_1 \to A_1/\mathrm{Ker}\,(f)$ is the canonical projection to the quotient, $\pi(a) = a + \mathrm{Ker}\,(f)$.

Proof. The assertions (1°) and (2°) are evident. For (3°), take $\bar{f}$ to be the injective linear map that one gets from the isomorphism theorem of vector spaces applied to the present case, and notice that it is an algebra homomorphism since

$$\bar{f}\Big((a + \mathrm{Ker}\, f)(b + \mathrm{Ker}\, f)\Big) \; = \; \bar{f}(ab + \mathrm{Ker}\, f) \; = \; f(ab) \; = \; f(a)\, f(b) \; = \; \bar{f}(a + \mathrm{Ker}\, f)\; \bar{f}(b + \mathrm{Ker}\, f).$$

□

The definition of a representation is analogous to the one for groups:

Definition 3. *For A an algebra and V a vector space, a* representation *of A in V is an algebra homomorphism $\rho : A \to \mathrm{End}(V)$.*

In such a case we often call V an A-module (more precisely, a left A-module) and write $a.v = \rho(a)v$ for $a \in A$, $v \in V$.

Given an algebra $A = (A, \circ, 1_A)$, the opposite algebra A^{op} is the algebra $A^{\mathrm{op}} = (A, \circ^{\mathrm{op}}, 1_A)$ with the product operation reversed

$$a \;\circ^{\mathrm{op}}\; b \; = \; b \circ a \qquad \text{for all } a, b \in A.$$

Representations of the opposite algebra correspond to right A-modules, that is, vector spaces V with a right multiplication by elements of A which satisfy $v.1_A = v$ and $(v.a).b = v.(ab)$ for all $v \in V$, $a, b \in A$.

Subrepresentations (submodules), irreducible representations (simple modules), quotient representations (quotient modules) and direct sums of representations (direct sums of modules) are defined in the same way as before. For representations of algebras in complex vector spaces, Schur's lemma continues to hold and the proof is the same as before.

The most obvious example of a representation of an algebra is the algebra itself:

Example 5. *The algebra A is a left A-module by $a.b = ab$ (for all a in the algebra A and b in the module A), and a right A-module by the same formula (then we should read that a is in the module A and b in the algebra A).*

Also the dual of an algebra is easily equipped with a representation structure.

Example 6. *The dual A^* becomes a left A-module if we define $a.f \in A^*$ by*

$$\langle a.f, x \rangle = \langle f, xa \rangle$$

for $f \in A^*$, $a, x \in A$. Indeed, the property $1_A.f = f$ is evident and we check

$$\langle a.(b.f), x \rangle = \langle b.f, xa \rangle = \langle f, (xa)b \rangle = \langle f, x(ab) \rangle = \langle (ab).f, x \rangle.$$

Similarly, the dual becomes a right A-module by the definition

$$\langle f.a, x \rangle = \langle f, ax \rangle$$

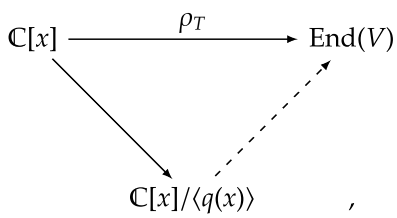
Example 7. *Representations of a group G correspond to representations of the group algebra $\mathbb{C}[G]$. Indeed, given a representation of the group, $\rho_G : G \to \mathrm{GL}(V)$, there is a unique linear extension of it from the values on the basis vectors, $e_g \mapsto \rho_G(g) \in \mathrm{GL}(V) \subset \mathrm{End}(V)$. The other way around, given an algebra representation $\rho_A : \mathbb{C}[G] \to \mathrm{End}(V)$, we observe that $\rho_A(e_g)$ is an invertible linear map with inverse $\rho_A(e_{g^{-1}})$, so we set $g \mapsto \rho_A(e_g)$ to define a representation of the group. Both ways the homomorphism property of the constructed map obviously follows from the homomorphism property of the original map.*

Example 8. *Let V be a vector space over $\mathbb{K}$ and $T \in \mathrm{End}(V)$ a linear map of V into itself. Since the polynomial algebra $\mathbb{K}[x]$ is the free (commutative) algebra with one generator x, there exists a unique algebra morphism $\rho_T : \mathbb{K}[x] \to \mathrm{End}(V)$ such that $x \mapsto T$, namely*

$$\rho_T\Big(c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n\Big) = c_0 + c_1 T + c_2 T^2 + \cdots + c_n T^n.$$

Thus any endomorphism T of a vector space defines a representation ρ_T of the algebra $\mathbb{K}[x]$. Likewise, any $n \times n$ matrix with entries in $\mathbb{K}$, interpreted as an endomorphism of $\mathbb{K}^n$, defines a representation of the polynomial algebra.

Example 9. *Let V be a complex vector space, and $T \in \mathrm{End}(V)$ as above and let $q(x) \in \mathbb{C}[x]$ be a polynomial. Consider the algebra $\mathbb{C}[x]/\langle q(x) \rangle$, where $\langle q(x) \rangle$ is the ideal generated by $q(x)$. The above representation map $\rho_T : \mathbb{C}[x] \to \mathrm{End}(V)$ factors through the quotient algebra $\mathbb{C}[x]/\langle q(x) \rangle$*



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if and only if the ideal $\langle q(x) \rangle$ is contained in the ideal $\mathrm{Ker}\ \rho_T$. The ideal $\mathrm{Ker}\ \rho_T$ is generated by the minimal polynomial of T (recall that the polynomial algebra is a principal ideal domain: any ideal is generated by one element, a lowest degree nonzero polynomial contained in the ideal). Thus the above factorization through quotient is possible if and only if the minimal polynomial of T divides $q(x)$. We conclude that the representations of the algebra $\mathbb{C}[x]/\langle q(x) \rangle$ correspond to endomorphisms whose minimal polynomial divides $q(x)$ — or equivalently, to endomorphisms T such that $q(T) = 0$.

The Jordan decomposition of complex matrices gives a direct sum decomposition of this representation with summands corresponding to the invariant subspaces of each Jordan block. The direct summands are indecomposable (not themself expressible as direct sum of two proper subreprentations) but those corresponding to blocks of size more than one are not irreducible (they contain proper subrepresentations, for example the one dimensional eigenspace within the block). We see that whenever $q(x)$ has roots of multiplicity greater than one, there are representations of the algebra $\mathbb{C}[x]/\langle q(x) \rangle$ which are not completely reducible.

Another definition of algebra

In our definitions of algebras, coalgebras and Hopf algebras we will from here on take the ground field to be the field $\mathbb{C}$ of complex numbers, although much of the theory could be developed for other fields, too.

The following "tensor flip" will be used occasionally. For V and W vector spaces, let us denote by $S_{V,W}$ the linear map that switches the components

$$S_{V,W} : V \otimes W \to W \otimes V \qquad \text{such that} \qquad S_{V,W}(v \otimes w) = w \otimes v \qquad \forall\, v \in V, w \in W.$$

By the bilinearity axiom for algebras, the product could be factorized through $A \otimes A$, namely there exists a linear map $\mu : A \otimes A \to A$ such that

$$\mu(a \otimes b) \; = \; a\,b \qquad \forall\, a, b \in A.$$

We can also encode the unit in a linear map

$$\eta : \mathbb{C} \to A \qquad \lambda \mapsto \lambda\, 1_A.$$

The axioms of associativity and unitality then read

$$\mu \circ (\mu \otimes \mathrm{id}_A) \; = \; \mu \circ (\mathrm{id}_A \otimes \mu) \tag{H1}$$

$$\mu \circ (\eta \otimes \mathrm{id}_A) \; = \; \mathrm{id}_A \; = \; \mu \circ (\mathrm{id}_A \otimes \eta), \tag{H2}$$

where (H1) expresses the equality of two maps $A \otimes A \otimes A \to A$, when we make the usual identifications

$$(A \otimes A) \otimes A \cong A \otimes A \otimes A \cong A \otimes (A \otimes A)$$

and (H2) expresses the equality of three maps $A \to A$, when we make the usual identifications

$$\mathbb{C} \otimes A \cong A \cong A \otimes \mathbb{C}.$$

We take this as our definition (it is equivalent to the standard definition).

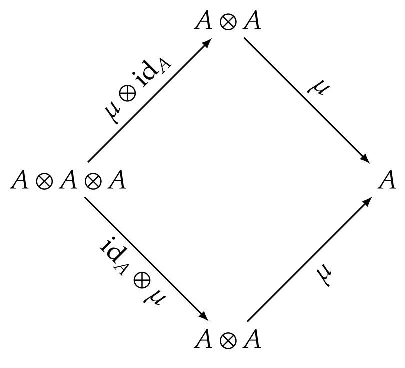
Definition 4. *An (associative unital) algebra is a triple (A, μ, η), where A is a vector space and*

$$\mu : A \otimes A \to A \qquad \eta : \mathbb{C} \to A$$

are linear maps, such that "associativity" (H1) and "unitality" (H2) hold.

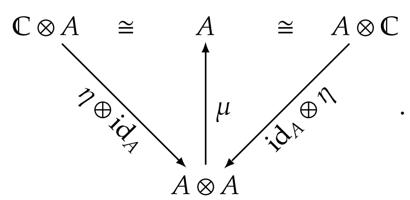
Example 10. *If (A, μ, η) is an algebra, then setting $\mu^{\mathrm{op}} = \mu \circ S_{A,A}$, i.e. $\mu^{\mathrm{op}}(a \otimes b) = b\,a$, one obtains the opposite algebra $A^{\mathrm{op}} = (A, \mu^{\mathrm{op}}, \eta)$. An algebra is called commutative if $\mu^{\mathrm{op}} = \mu$.*

The axiom of associativity can also be summarized by the following commutative diagram



(H1)

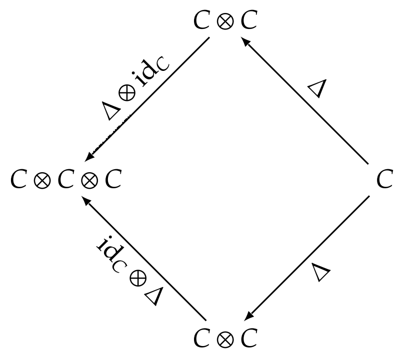
and unitality by



(H2)

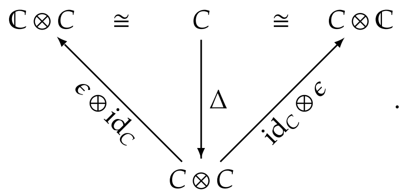
Coalgebras

A coalgebra is defined by reversing the directions of all arrows in the commutative diagrams defining algebras. Namely, we impose an axiom of "coassociativity"



(H1′)

and "counitality"



. (H2′)

Definition 5. *A coalgebra is a triple (C, Δ, ϵ), where C is a vector space and*

$$\Delta : C \to C \otimes C \qquad \epsilon : C \to \mathbb{C}$$

are linear maps such that "coassociativity" (H1′) and "counitality" (H2′) hold. The maps Δ and ϵ are called coproduct and counit, respectively.

The axioms for coalgebras can also be written as

$$(\Delta \otimes \mathrm{id}_C) \circ \Delta \;=\; (\mathrm{id}_C \otimes \Delta) \circ \Delta \qquad \text{(H1′)}$$

$$(\epsilon \otimes \mathrm{id}_C) \circ \Delta \;=\; \mathrm{id}_C \;=\; (\mathrm{id}_C \otimes \epsilon) \circ \Delta. \qquad \text{(H2′)}$$

Example 11. *If (C, Δ, ϵ) is a coalgebra, then with the opposite coproduct $\Delta^{\mathrm{cop}} = S_{C,C} \circ \Delta$ one obtains the (co-)opposite coalgebra $C^{\mathrm{cop}} = (C, \Delta^{\mathrm{cop}}, \epsilon)$. A coalgebra is called cocommutative if $\Delta^{\mathrm{cop}} = \Delta$.*

Sweedler's sigma notation

For practical computations with coalgebras it's important to have manageable notational conventions. We will follow what is known as the Sweedler's sigma notation. By usual properties of the tensor product, we can for any $a \in C$ write the coproduct of a as a linear combination of simple tensors

$$\Delta(a) \;=\; \sum_{j=1}^{n} a'_j \otimes a''_j .$$

In such expressions the choice of simple tensors, or the choice of $a'_j, a''_j \in C$, or even the number n of terms, are of course not unique! It is nevertheless convenient to keep this property in mind and use the notation

$$\Delta(a) \;=\; \sum_{(a)} a_{(1)} \otimes a_{(2)}$$

to represent any of the possible expressions for $\Delta(a) \in C \otimes C$. Likewise, when $a \in C$, and we write some expression involving a sum $\sum_{(a)}$ and bilinear dependency on $a_{(1)}$ and $a_{(2)}$, it is to be interpreted so that any linear combination of simple tensors that gives the coproduct of a could be used. For example, if $g : C \to V$ and $h : C \to W$ are linear maps, then

$$\sum_{(a)} g(a_{(1)}) \otimes h(a_{(2)}) \qquad \text{represents} \qquad (g \otimes h)(\Delta(a)) \in V \otimes W.$$

The opposite coproduct of Example 11 is written in this notation as

$$\Delta^{\mathrm{cop}}(a) \;=\; S_{C,C}(\Delta(a)) \;=\; \sum_{(a)} a_{(2)} \otimes a_{(1)}.$$

Another example is the counitality axiom, which reads

$$\sum_{(a)} \epsilon(a_{(1)})\, a_{(2)} \;=\; a \;=\; \sum_{(a)} \epsilon(a_{(2)})\, a_{(1)}. \tag{H2'}$$

The coassociativity axiom states that

$$\sum_{(a)} \sum_{(a_{(1)})} (a_{(1)})_{(1)} \otimes (a_{(1)})_{(2)} \otimes a_{(2)} \;=\; \sum_{(a)} \sum_{(a_{(2)})} a_{(1)} \otimes (a_{(2)})_{(1)} \otimes (a_{(2)})_{(2)}. \tag{H1'}$$

By a slight abuse of notation we write the above quantity as $\sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$, and more generally we write the $n-1$-fold coproduct as

$$(\Delta \otimes \mathrm{id}_C \otimes \cdots \otimes \mathrm{id}_C) \circ \;\ldots\; \circ (\Delta \otimes \mathrm{id}_C) \circ \Delta(a) \;=\; \sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes \cdots \otimes a_{(n)}.$$

So when reading an expression involving the Sweedler's notation $\sum_{(a)}$, one should always check what is the largest subscript index of the $a_{(j)}$ in order to know how many coproducts are successively applied to a — note, however, that by coassociativity it doesn't matter to which components we apply the coproducts.

Subcoalgebras, coideals, quotient coalgebras and isomorphism theorem

As for other algebraic structures, maps that preserve the structure are called homomorphisms, and one can define substructures and quotient structures, and one has an isomorphism theorem.

Definition 6. *Let $(C_j, \Delta_j, \epsilon_j)$, $j = 1, 2$, be two coalgebras. A homomorphism of coalgebras is linear map $f : C_1 \to C_2$ which preserves the coproduct and counit in the following sense*

$$\Delta_2 \circ f \;=\; (f \otimes f) \circ \Delta_1 \qquad \text{and} \qquad \epsilon_2 \circ f \;=\; \epsilon_1.$$

Definition 7. *For $C = (C, \Delta, \epsilon)$ a coalgebra, a vector subspace $C' \subset C$ is called a <u>subcoalgebra</u> if*

$$\Delta(C') \;\subset\; C' \otimes C'.$$

A vector subspace $J \subset C$ is called a <u>coideal</u> if

$$\Delta(J) \;\subset\; J \otimes C + C \otimes J \qquad \text{and} \qquad \epsilon\big|_J \;=\; 0.$$

For $J \subset C$ a coideal, the quotient vector space C/J becomes a coalgebra by the coproduct and counit

$$\Delta_{C/J}(a + J) \;=\; \sum_{(a)} (a_{(1)} + J) \otimes (a_{(2)} + J) \qquad \text{and} \qquad \epsilon_{C/J}(a + J) = \epsilon(a),$$

whose well-definedness is again due to the coideal properties of J.

The isomorphism theorem for coalgebras is the following (unsurprising) statement.

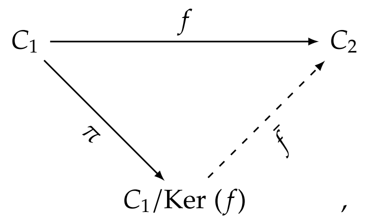
Theorem 2. *Let C_1 and C_2 be coalgebras and $f : C_1 \to C_2$ a homomorphism of coalgebras. Then*

1°) $\mathrm{Im}\,(f) := f(C_1) \subset C_2$ *is a subcoalgebra.*

2°) $\mathrm{Ker}\,(f) := f^{-1}(\{0\}) \subset C_1$ *is a coideal.*

3°) The quotient coalgebra $C_1/\mathrm{Ker}\,(f)$ *is isomorphic to* $\mathrm{Im}\,(f)$.

More precisely, there exists an injective homomorphism of coalgebras $\bar{f} : C_1/\mathrm{Ker}\,(f) \to C_2$ *such that the following diagram commutes*



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where $\pi : C_1 \to C_1/\mathrm{Ker}\,(f)$ *is the canonical projection to the quotient,* $\pi(a) = a + \mathrm{Ker}\,(f)$.

Bialgebras and Hopf algebras

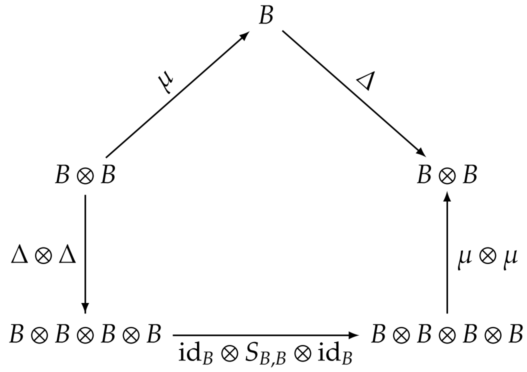
Definition 8. *A bialgebra is a quintuple* $(B, \mu, \Delta, \eta, \epsilon)$ *where* B *is a vector space and*

$$\mu : B \otimes B \to B \qquad \Delta : B \to B \otimes B$$
$$\eta : \mathbb{C} \to B \qquad \epsilon : B \to \mathbb{C}$$

are linear maps so that (B, μ, η) *is an algebra,* (B, Δ, ϵ) *is a coalgebra and the following further axioms hold*

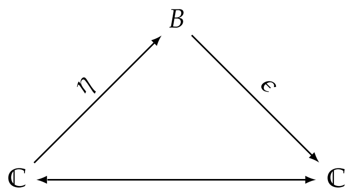
$$\Delta \circ \mu = (\mu \otimes \mu) \circ (\mathrm{id}_B \otimes S_{B,B} \otimes \mathrm{id}_B) \circ (\Delta \otimes \Delta) \tag{H4}$$
$$\Delta \circ \eta = \eta \otimes \eta \tag{H5}$$
$$\epsilon \circ \mu = \epsilon \otimes \epsilon \tag{H5'}$$
$$\epsilon \circ \eta = \mathrm{id}_{\mathbb{C}}. \tag{H6}$$

The following commutative diagrams visualize the new axioms:



(H4)

(H5 and H5′)



(H6)

In the exercises it is checked that the axioms (H4), (H5), (H5′), (H6) state alternatively that Δ and ϵ are homomorphisms of algebras, or that μ and η are homomorphisms of coalgebras. We will soon also motivate this definition with properties of representations.

Hopf algebras have one more structural map and one more axiom:

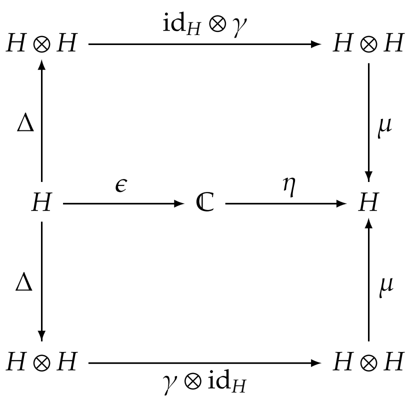
Definition 9. *A Hopf algebra is a sextuple $(H, \mu, \Delta, \eta, \epsilon, \gamma)$, where H is a vector space and*

$$\mu : H \otimes H \to H \qquad \Delta : H \to H \otimes H$$
$$\eta : \mathbb{C} \to H \qquad \epsilon : H \to \mathbb{C}$$
$$\gamma : H \to H$$

are linear maps such that $(H, \mu, \Delta, \eta, \epsilon)$ is a bialgebra and the following further axiom holds

$$\mu \circ (\gamma \otimes \mathrm{id}_H) \circ \Delta \;=\; \eta \circ \epsilon \;=\; \mu \circ (\mathrm{id}_H \otimes \gamma) \circ \Delta. \qquad \text{(H3)}$$

The map $\gamma : H \to H$ is called antipode. The corresponding commutative diagram is



(H3)

To get familiar with the Sweedler's sigma notation we rewrite the axiom concerning the antipode as follows

$$\sum_{(a)} \gamma(a_{(1)})\, a_{(2)} \;=\; \epsilon(a)\, 1_H \;=\; \sum_{(a)} a_{(1)}\, \gamma(a_{(2)}) \qquad \forall\, a \in H, \tag{H3}$$

where $1_H = \eta(1)$ is the unit of the algebra (H, μ, η) and we use the usual notation for products in the algebra, $a\,b := \mu(a \otimes b)$.

Example 12. *The group algebra $\mathbb{C}[G]$ of a group G becomes a Hopf algebra with the definitions*

$$\Delta(e_g) \;=\; e_g \otimes e_g\,, \qquad \epsilon(e_g) \;=\; 1\,, \qquad \gamma(e_g) \;=\; e_{g^{-1}} \qquad \text{(extended linearly).}$$

We call this Hopf algebra the Hopf algebra of the group G, and continue to use the notation $\mathbb{C}[G]$ for it.

Example 13. *The algebra of polynomials $\mathbb{C}[x]$ becomes a Hopf algebra with the definitions*

$$\Delta(x^n) \;=\; \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}\,, \qquad \epsilon(x^n) \;=\; \delta_{n,0}\,, \qquad \gamma(x^n) \;=\; (-1)^n\, x^n \qquad \text{(extended linearly),}$$

where

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

are the binomial coefficients, and we've used the Kronecker delta symbol

$$\delta_{n,m} \;=\; \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}.$$

Motivated by the above examples, we give names to some elements whose coproduct resembles one of the two examples.

Definition 10. *Let (C, Δ, ϵ) be a coalgebra. A non-zero element $a \in C$ is said to be grouplike if $\Delta(a) = a \otimes a$. Let $(B, \mu, \Delta, \eta, \epsilon)$ be a bialgebra. A non-zero element $x \in B$ is said to be primitive if $\Delta(x) = x \otimes 1_B + 1_B \otimes x$.*

All the basis vectors e_g, $g \in G$, in the Hopf algebra $\mathbb{C}[G]$ of a group G are grouplike. Any scalar multiple of the indeterminate x in the binomial Hopf algebra $\mathbb{C}[x]$ is primitive. Here's one more obvious example:

Example 14. *If $(B, \mu, \eta, \Delta, \epsilon)$ is a bialgebra, then the unit $1_B = \eta(1) \in B$ is grouplike by the property (H5).*

Lemma 3. *Let $(B, \mu, \eta, \Delta, \epsilon)$ be a bialgebra. Then*

- *for any grouplike element $a \in B$ we have $\epsilon(a) = 1$*
- *for any primitive element $x \in B$ we have $\epsilon(x) = 0$.*

If B furthermore admits a Hopf algebra structure with the antipode $\gamma : B \to B$ then

- *any grouplike element $a \in B$ is invertible and we have $\gamma(a) = a^{-1}$*
- *for any primitive element $x \in B$ we have $\gamma(x) = -x$.*

Motivation for the definitions from representations

Recall that for a finite group we were able not only to take direct sums of representations, but also we made the tensor product of representations a representation, the one dimensional vector space a trivial representation, and the dual of a representation a representation.

Suppose now A is an algebra and $\rho_V : A \to \mathrm{End}(V)$ and $\rho_W : A \to \mathrm{End}(W)$ are representations of A in V and W, respectively. Taking direct sums of the representations works just like before: we set

$$a.(v + w) \;=\; \rho_V(v) + \rho_W(w) \qquad \text{for all } v \in V \subset V \oplus W \text{ and } w \in W \subset V \oplus W.$$

It seems natural to ask how to take tensor products of representations of algebras, and the answer is that one needs some extra structure. In fact, the coproduct $\Delta : A \to A \otimes A$ with the axioms (H4) and (H5) precisely guarantees that the formula $(\rho_V \otimes \rho_W) \circ \Delta$ defines a representation of A on $V \otimes W$. With Sweedler's sigma notation this reads

$$a.(v \otimes w) \;=\; \sum_{(a)} (a_{(1)}.v) \otimes (a_{(2)}.w) \qquad \text{for } v \in V,\ w \in W. \tag{3.1}$$

In particular, using the coproduct of Example 12 this definition coincides with the definition of tensor product representation we gave for groups.

Can we make the ground field $\mathbb{C}$ a trivial representation? Indeed, when we interpret $\mathrm{End}(\mathbb{C}) \cong \mathbb{C}$, identifying a linear map $\mathbb{C} \to \mathbb{C}$ with its sole eigenvalue, a map $\epsilon : A \to \mathbb{C}$ becomes a one dimensional representation if and only if the axioms (H5′) and (H6) hold. So when we have a counit ϵ we set

$$a.\lambda \;=\; \epsilon(a)\,\lambda \in \mathbb{C} \qquad \text{for } \lambda \in \mathbb{C}, \tag{3.2}$$

and call this the trivial representation. Again, using the counit of Example 12, the trivial representation of a group is what we defined it to be before.

Exercise 1. *Check that the formulas (3.1) and (3.2) define representations if we assume the axioms mentioned. Check also that with the Hopf algebra structure on $\mathbb{C}[G]$ given in Example 12, these definitions agree with the corresponding representations of groups.*

How about duals then? For any representation V we'd like to make $V^* = \mathrm{Hom}(V, \mathbb{C})$ a representation. Recall also that under finite dimensionality assumption $\mathrm{Hom}(V, W) \cong W \otimes V^*$, so since we already know how to handle tensor product representations with the coproduct, we might in fact hope to make the space of linear maps between representations a representation. When we have an antipode satisfying (H3), the formula

$$a.T \;=\; \sum_{(a)} \rho_W(a_{(1)}) \circ T \circ \rho_V(\gamma(a_{(2)}))$$

turns out to work, as we will see a bit later. Again, the antipode of Example 12 leads to the definitions we gave for groups.

Although we have given a representation theoretic interpretation for the coproduct Δ, the counit ϵ, and the antipode γ, so far we didn't use axioms (H1′) and (H2′) of a coalgebra. It is easy to see, however, that the canonical linear isomorphism between the triple tensor products

$$(V_1 \otimes V_2) \otimes V_3 \qquad \text{and} \qquad V_1 \otimes (V_2 \otimes V_3)$$

becomes an isomorphism of representations with the definition (3.1) when coassociativity (H1′) is imposed. Likewise, the canonical identifications of V with

$$V \otimes \mathbb{C} \qquad \text{and} \qquad \mathbb{C} \otimes V$$

become isomorphisms of representations with the definition (3.2) when counitality (H2′) is imposed.

Thus we see that the list of nine axioms (H1), (H1′), (H2), (H2′), (H3), (H4), (H5), (H5′), (H6) is very natural in view of standard operations that we want to perform for representations.

One more remark is in order: the "flip"

$$S_{V,W} : V \otimes W \to W \otimes V \qquad v \otimes w \mapsto w \otimes v$$

gives a rather natural vector space isomorphism between $V \otimes W$ and $W \otimes V$. With the definition (3.1), it would be an isomorphism of representations if we required the coproduct to be cocommutative, i.e. that the coproduct Δ is equal to the opposite coproduct $\Delta^{\mathrm{cop}} := S_{A,A} \circ \Delta$. However, we choose *not* to require cocommutativity in general — in fact the most interesting examples of Hopf algebras are certain quantum groups, where instead of "flipping" the factors of tensor product by $S_{V,W}$ we can do "braiding" on the factors. We will return to this point later on in the course.

The dual of a coalgebra

When $f : V \to W$ is a linear map, its transpose is the linear map $f^* : W^* \to V^*$ given by

$$\langle f^*(\varphi), v \rangle \;=\; \langle \varphi, f(v) \rangle \qquad \text{for all } \varphi \in W^*, v \in V.$$

Recall also that we have the inclusion $V^* \otimes W^* \subset (V \otimes W^*)$ with the interpretation

$$\langle \psi \otimes \varphi, v \otimes w \rangle \;=\; \langle \psi, v \rangle \, \langle \varphi, w \rangle \qquad \text{for } \psi \in V^*, \varphi \in W^*, v \in V, w \in W,$$

and observe that the dual of the ground field can be naturally identified with the ground field itself

$$\mathbb{C}^* \;\cong\; \mathbb{C} \qquad \text{via} \qquad \mathbb{C}^* \;\ni\; \phi \;\leftrightarrow\; \langle \phi, 1 \rangle \;\in\; \mathbb{C}.$$

Theorem 4. *Let C be a coalgebra, with coproduct $\Delta : C \to C \otimes C$ and counit $\epsilon : C \to \mathbb{C}$. Set $A = C^*$ and*

$$\mu \;=\; \Delta^*\big|_{C^* \otimes C^*} \;:\; A \otimes A \to A \qquad , \qquad \eta \;=\; \epsilon^* : \mathbb{C} \to A.$$

Then (A, μ, η) is an algebra.

Proof. Denote $1_A = \eta(1)$. Compute for $\varphi \in C^* = A$ and $c \in C$, using (H2′) in the last step,

$$\langle \varphi \, 1_A, c \rangle \;=\; \langle \varphi \otimes 1_A, \Delta(c) \rangle \;=\; \sum_{(c)} \langle \varphi, c_{(1)} \rangle \, \langle 1_A, c_{(2)} \rangle \;=\; \langle \varphi, \sum_{(c)} c_{(1)} \epsilon(c_{(2)}) \rangle \;=\; \langle \varphi, c \rangle$$

to obtain $\varphi \, 1_A = \varphi$. Similarly one proves $1_A \, \varphi = \varphi$ and gets unitality for A. Associativity of $\mu = \Delta^*$ is also easy to show using the coassociativity (H2′) of Δ. □

Convolution algebras

One of the main goals of this section is to prove the following facts about the antipode.

Theorem 5. *Let $H = (H, \mu, \Delta, \eta, \epsilon, \gamma)$ be a Hopf algebra.*

- *(!) The antipode γ is unique.*
- *(i) The map $\gamma : H \to H$ is a homomorphism of algebras from $H = (H, \mu, \eta)$ to $H^{\mathrm{op}} = (H, \mu^{\mathrm{op}}, \eta)$.*
- *(ii) The map $\gamma : H \to H$ is a homomorphism of coalgebras from $H = (H, \Delta, \epsilon)$ to $H^{\mathrm{cop}} = (H, \Delta^{\mathrm{cop}}, \epsilon)$.*

In other words the property (i) says that we have $\gamma(1_H) = 1_H$ and

$$\gamma(a\, b) \;=\; \gamma(b)\, \gamma(a) \qquad \forall a, b \in H.$$

The property (ii) says that we have

$$\gamma(\Delta(a)) \;=\; \sum_{(a)} \gamma(a_{(2)})\, \gamma(a_{(1)}) \qquad \text{and} \qquad \epsilon(\gamma(a)) \;=\; \epsilon(a) \qquad \forall a \in H.$$

Definition 11. *Let $C = (C, \Delta, \epsilon)$ be a coalgebra and $A = (A, \mu, \eta)$ an algebra. For f, g linear maps $C \to A$ define the convolution product of f and g as the linear map*

$$f \star g \;=\; \mu \circ (f \otimes g) \circ \Delta \;:\; C \to A,$$

and the convolution unit $1_\star$ as the linear map

$$1_\star \;=\; \eta \circ \epsilon \;:\; C \to A.$$

The convolution algebra associated with C and A is the vector space $\mathrm{Hom}(C, A)$ *equipped with product $\star$ and unit $1_\star$. The convolution algebra of a bialgebra $B = (B, \mu, \Delta, \eta, \epsilon)$ is the convolution algebra associated with the coalgebra (B, Δ, ϵ) and the algebra (B, μ, η), and the convolution algebra of a Hopf algebra is defined similarly.*

Proposition 6. *The convolution algebra is an associative unital algebra.*

Sketch of a proof. Associativity for the convolution algebra follows easily from the associativity of A and coassociativity of C, and unitality of the convolution algebra follows easily from the unitality of A and counitality of C. □

Concolution algebras have applications for example in combinatorics. For now, we will use them to prove properties of the antipode.

Proof of Theorem 5. Let us first prove the uniqueness (!). By (H3), the antipode $\gamma \in \mathrm{Hom}(H, H)$ is the two-sided convolutive inverse of $\mathrm{id}_H \in \mathrm{Hom}(H, H)$ in the convolution algebra of the Hopf algebra H, that is we have

$$\gamma \star \mathrm{id}_H = 1_\star = \mathrm{id}_H \star \gamma.$$

In an associative algebra a left inverse has to coincide with a right inverse if both exist. Indeed suppose that γ' would also satisfy (H3) so that in particular $\mathrm{id}_H \star \gamma' = 1_\star$. Then we compute

$$\gamma \;=\; \gamma \star 1_\star \;=\; \gamma \star (\mathrm{id}_H \star \gamma') \;=\; (\gamma \star \mathrm{id}_H) \star \gamma' \;=\; 1_\star \star \gamma' \;=\; \gamma'.$$

Then let us prove (i): the antipode is a homomorphism of algebras to the opposite algebra. We must show that the antipode preserves the unit, $\gamma \circ \eta = \eta$, and that it reverses the product, $\gamma \circ \mu = \mu^{\mathrm{op}} \circ (\gamma \otimes \gamma)$. Preserving unit is easily seen: recall that 1_H is grouplike, $\Delta(1_H) = 1_H \otimes 1_H$ and then apply (H3) to 1_H to see that

$$1_H \overset{\text{(H3)}}{=} \big(\mu \circ (\gamma \otimes \mathrm{id}_H)\big)(1_H \otimes 1_H)) \;=\; \gamma(1_H)\, 1_H \overset{\text{(H2)}}{=} \gamma(1_H).$$

Now consider the convolution algebra $\mathrm{Hom}(H \otimes H, H)$ associated with the coalgebra $H \otimes H$ with coproduct and counit as follows

$$\begin{aligned} \Delta_2(a \otimes b) \;&=\; \sum_{(a),(b)} a_{(1)} \otimes b_{(1)} \otimes a_{(2)} \otimes b_{(2)} \;=\; \Big((\mathrm{id}_H \otimes S_{H,H} \otimes \mathrm{id}_H) \circ (\Delta \otimes \Delta)\Big)(a \otimes b) \\ \epsilon_2(a \otimes b) \;&= \epsilon(a)\, \epsilon(b) \end{aligned}$$

and with the algebra $H = (H, \mu, \eta)$. Note that we can write $\Delta_2 = (\mathrm{id}_H \otimes S_{H,H} \otimes \mathrm{id}_H) \circ (\Delta \otimes \Delta)$ and $\epsilon_2 = \epsilon \otimes \epsilon$. We will show (a) that $\mu \in \mathrm{Hom}(H \otimes H, H)$ has a right convolutive inverse $\gamma \circ \mu^{\mathrm{op}}$, and (b) that μ has a left convolutive inverse $\mu \circ (\gamma \otimes \gamma)$. To prove (a), compute for $a, b \in H$

$$\begin{aligned} \mu \star (\gamma \circ \mu) \;&= \mu \circ \big(\mu \otimes (\gamma \circ \mu)\big) \circ \Delta_2 \\ &= \mu \circ (\mathrm{id}_H \otimes \gamma) \circ (\mu \otimes \mu) \circ (\mathrm{id}_H \otimes S_{H,H} \otimes \mathrm{id}_H) \circ (\Delta \otimes \Delta) \\ &\overset{\text{(H4)}}{=} \mu \circ (\mathrm{id}_H \otimes \gamma) \circ \Delta \circ \mu \\ &\overset{\text{(H3)}}{=} \eta \circ \epsilon \circ \mu \\ &\overset{\text{(H5')}}{=} \eta \circ (\epsilon \otimes \epsilon) \;=\; \eta \circ \epsilon_2 \;=\; 1_\star \end{aligned}$$

To prove (b), compute in the Sweedler's sigma notation

$$\Big(\big(\mu \circ S_{H,H} \circ (\gamma \otimes \gamma)\big) \star \mu\Big)(a \otimes b) \;=\; \sum_{(a),(b)} \big(\gamma(b_{(1)})\,\gamma(a_{(1)})\big)\,\big(a_{(2)}\,b_{(2)}\big)$$
$$\overset{\text{(H3) for } a}{=} \; \epsilon(a) \sum_{(b)} \gamma(b_{(1)})\, 1_H\, b_{(2)}$$
$$\overset{\text{(H3) for } b}{=} \; \epsilon(a)\epsilon(b)\; 1_H,$$

which is the value of $1_\star = \eta \circ \epsilon_2$ on the element $a \otimes b$. Now a right inverse of μ has to coincide with a left inverse of μ, so we get

$$\gamma \circ \mu \;=\; \mu \circ S_{H,H} \circ (\gamma \otimes \gamma),$$

as we wanted.

We leave it as an exercise for the reader to prove (ii) by finding appropriate formulas for right and left inverses of Δ in the convolution algebra $\mathrm{Hom}(H, H \otimes H)$. □

Corollary 7. *For $H = (H, \mu, \Delta, \eta, \epsilon, \gamma)$ a Hopf algebra, V and W representations of (H, μ, η), the space $\mathrm{Hom}(V, W)$ of linear maps between the representations becomes a representation by the formula*

$$\big(a.T\big)(v) \;=\; \sum_{(a)} a_{(1)}.\Big(T(\gamma(a_{(2)}).v)\Big) \qquad \text{for } a \in H,\ T \in \mathrm{Hom}(V, W),\ v \in V.$$

Proof. The property $1_H.T = T$ is obvious in view of $\Delta(1_H) = 1_H \otimes 1_H$ and $\gamma(1_H) = 1_H$. Using the facts that $\gamma : H \to H^{\mathrm{op}}$ and $\Delta : H \to H \otimes H$ are homomorphisms of algebras, we also check

$$\big(a.(b.T)\big)(v) \;=\; \sum_{(a)} a_{(1)}.\Big((b.T)(\gamma(a_{(2)}).v)\Big) \;=\; \sum_{(a),(b)} a_{(1)}.b_{(1)}.\Big(T(\gamma(b_{(2)})\gamma(a_{(2)}).v)\Big)$$
$$\overset{\gamma \text{ homom.}}{=} \sum_{(a),(b)} (a_{(1)}b_{(1)}).\Big(T(\gamma(a_{(2)}b_{(2)}).v)\Big) \overset{\Delta \text{ homom.}}{=} \sum_{(ab)} (ab)_{(1)}.\Big(T(\gamma((ab)_{(2)}).v)\Big)$$
$$= \Big((ab).T\Big)(v).$$

□

Corollary 8. *Suppose that $H = (H, \mu, \Delta, \eta, \epsilon, \gamma)$ is a Hopf algebra which is either commutative or cocommutative. Then the antipode is involutive, that is $\gamma \circ \gamma = \mathrm{id}_H$.*

Proof. Assume that A is commutative. Now, since γ is a morphism of algebras $A \to A^{\mathrm{op}}$ we have

$$\begin{aligned}
\gamma^2 \star \gamma \;&= \mu \circ (\gamma^2 \otimes \gamma) \circ \Delta \\
&= \gamma \circ \mu^{\mathrm{op}} \circ (\gamma \otimes \mathrm{id}_A) \circ \Delta \\
&= \gamma \circ \mu \circ (\gamma \otimes \mathrm{id}_A) \circ \Delta \\
&= \gamma \circ \eta \circ \epsilon \;=\; \eta \circ \epsilon \;=\; 1_\star.
\end{aligned}$$

We conclude that γ^2 is a left inverse of γ in the convolution algebra (one could easily show that γ^2 is in fact a two-sided inverse). But id_A is a right (in fact two-sided) inverse of γ, and as usually in associative algebras we therefore get $\gamma^2 = \mathrm{id}_A$. The case of a cocommutative Hopf algebra is handled similarly. □

Above we showed that the antipode is an involution if the Hopf algebra is commutative or cocommutative. The cocommutativity $\Delta(x) = \Delta^{\mathrm{cop}}(x)$ will later be generalized a little: braided Hopf algebras have $\Delta(x)$ and $\Delta^{\mathrm{cop}}(x)$ conjugates of each other and we will show that the antipode is always invertible in such a case. We will also later show that the antipode of a finite dimensional Hopf algebra is always invertible. The following exercise characterizes invertibility of the antipode in terms of the existence of antipodes for the opposite and co-opposite bialgebras.

Exercise 2. *"Opposite and co-opposite bialgebras and Hopf algebras"*
Suppose that $A = (A, \mu, \Delta, \eta, \epsilon)$ is a bialgebra. Let $\mu^{\mathrm{op}} = \mu \circ S_{A,A}$ be the opposite product and $\Delta^{\mathrm{cop}} = S_{A,A} \circ \Delta$ be the (co-)opposite coproduct.

(a) Show that all of the following are bialgebras:

- *the opposite bialgebra $A^{\mathrm{op}} = (A, \mu^{\mathrm{op}}, \Delta, \eta, \epsilon)$*
- *the co-opposite bialgebra $A^{\mathrm{cop}} = (A, \mu, \Delta^{\mathrm{cop}}, \eta, \epsilon)$*
- *the opposite co-opposite bialgebra $A^{\mathrm{op,cop}} = (A, \mu^{\mathrm{op}}, \Delta^{\mathrm{cop}}, \eta, \epsilon)$.*

Suppose furthermore that $\gamma : A \to A$ satisfies (H3) so that $(A, \mu, \Delta, \eta, \epsilon, \gamma)$ is a Hopf algebra.

(b) Show that $A^{\mathrm{op,cop}} = (A, \mu^{\mathrm{op}}, \Delta^{\mathrm{cop}}, \eta, \epsilon, \gamma)$ is a Hopf algebra, called the the opposite co-opposite Hopf algebra.

(c) Show that the following conditions are equivalent

- *the opposite bialgebra A^{op} admits an antipode $\tilde{\gamma}$*
- *the co-opposite bialgebra A^{cop} admits an antipode $\tilde{\gamma}$*
- *the antipode $\gamma : A \to A$ is an invertible linear map, with inverse $\tilde{\gamma}$.*

Representative forms

Let $A = (A, \mu, \eta)$ be an algebra.

Suppose that V is a finite dimensional A-module, and that $u_1, u_2, \ldots, u_n$ is a basis of V. Note that for any $a \in A$ we can write $a.u_j = \sum_{i=1}^n \lambda_{i,j}\, u_i$ with $\lambda_{i,j} \in \mathbb{C}$, $i, j = 1, 2, \ldots, n$. The coefficients depend on a lineary and thus define elements of the dual $\lambda_{i,j} \in A^*$ called the representative forms of the A-module V with respect to the basis $u_1, u_2, \ldots, u_n$. The left multiplication of the basis vectors by elements of A now takes the form

$$a.u_j \;=\; \sum_{i=1}^n \langle \lambda_{i,j}, a\rangle\, u_i.$$

The A-module property gives

$$\sum_{i=1}^n \langle \lambda_{i,j}, ab\rangle\, u_i \;=\; (ab).v \;=\; a.(b.v) \;=\; \sum_{i,k=1}^n \langle \lambda_{i,k}, a\rangle\, \langle \lambda_{k,j}, b\rangle\, u_i,$$

that is

$$\langle \lambda_{i,j}, ab\rangle \;=\; \sum_{k=1}^n \langle \lambda_{i,k}, a\rangle\, \langle \lambda_{k,j}, b\rangle \qquad \text{for all } i, j = 1, 2, \ldots, n. \tag{3.3}$$

The restricted dual of algebras and Hopf algebras

Recall that for C a coalgebra, the dual space C^* becomes an algebra with the structural maps (product and unit) which are the transposes of the structural maps (coproduct and counit) of the coalgebra.

It then seems natural to ask whether the dual of an algebra $A = (A, \mu, \eta)$ is a coalgebra. When we take the transposes of the structural maps

$$\eta : \mathbb{C} \to A \qquad \text{and} \qquad \mu : A \otimes A \to A,$$

we get

$$\eta^* : \mathbb{C}^* \to A^*$$

which could serve as a counit when we identify $\mathbb{C}^* \cong \mathbb{C}$, but the problem is that the candidate for a coproduct

$$\mu^* : A^* \to (A \otimes A)^* \supset A^* \otimes A^*,$$

takes values in the space $(A \otimes A)^*$ which in general is larger than the second tensor power of the dual, $A^* \otimes A^*$. The cure to the situation is to restrict attention to the preimage of the second tensor power of the dual.

Definition 12. *The* <u>*restricted dual*</u> *of an algebra $A = (A, \mu, \eta)$ is the subspace $A^\circ \subset A^*$ defined as*

$$A^\circ = (\mu^*)^{-1}(A^* \otimes A^*).$$

Example 15. *Let V be a finite dimensional A-module with basis $u_1, u_2, \ldots, u_n$, and denote by $\lambda_{i,j} \in A^*$, $i, j = 1, 2, \ldots, n$, the representative forms. Then from Equation (3.3) we get for any $a, b \in A$*

$$\langle \mu^*(\lambda_{i,j}), a \otimes b \rangle = \langle \lambda_{i,j}, \mu(a \otimes b) \rangle = \langle \lambda_{i,j}, ab \rangle = \sum_{k=1}^{n} \langle \lambda_{i,k}, a \rangle \, \langle \lambda_{k,j}, b \rangle = \sum_{k=1}^{n} \langle \lambda_{i,k} \otimes \lambda_{k,j}, a \otimes b \rangle.$$

We conclude that

$$\mu^*(\lambda_{i,j}) = \sum_{k=1}^{n} \lambda_{i,k} \otimes \lambda_{k,j} \in A^* \otimes A^*, \tag{3.4}$$

and therefore $\lambda_{i,j} \in A^\circ$.

The example shows that all representative forms of finite dimensional A-modules are in the restricted dual, and we will soon see that the restricted dual is spanned by these.

The goal of this section is to prove the following results.

Theorem 9. *For $A = (A, \mu, \eta)$ an algebra, the restricted dual $(A^\circ, \mu^*|_{A^\circ}, \eta^*|_{A^\circ})$ is a coalgebra.*

Theorem 10. *For $H = (H, \mu, \Delta, \eta, \epsilon, \gamma)$ a Hopf algebra, the restricted dual*

$$(H^\circ, \Delta^*|_{H^\circ \times H^\circ}, \mu^*|_{H^\circ}, \epsilon^*, \eta^*|_{H^\circ}, \gamma^*|_{A^\circ})$$

is a Hopf algebra.

Before starting with the proofs, we need some preparations.

Lemma 11. *Let $A = (A, \mu, \eta)$ be an algebra and equip the dual A^* with the left A-module structure of Example 6. Then for any $f \in A^*$ we have*

$$f \in A^\circ \quad \Leftrightarrow \quad \dim(A.f) < \infty,$$

where $A.f \subset A^*$ is the submodule generated by f.

In other words, the elements of the restricted dual are precisely those that generate a finite dimensional submodule of A^*.

Remark 1. *Observe that $A^\circ = (\mu^*)^{-1}(A^* \otimes A^*) = ((\mu^{\mathrm{op}})^*)^{-1}(A^* \otimes A^*)$. Thus the analogous property holds for the right A-module structure of Example 6: we have $f \in A^\circ$ if and only if $f.A \subset A^*$ is finite dimensional.*

Proof of Lemma 11. Suppose first that $f \in A^\circ$, so that $\mu^*(f) = \sum_{i=1}^{n} g_i \otimes h_i$, for some $n \in \mathbb{N}$ and $g_i, h_i \in A^*$, $i = 1, 2, \ldots, n$. Then for any $a, x \in A$ we get

$$\begin{aligned} \big\langle a.f, x \big\rangle &= \big\langle f, xa \big\rangle = \big\langle f, \mu(x \otimes a) \big\rangle = \big\langle \mu^*(f), x \otimes a \big\rangle \\ &= \sum_{i=1}^{n} \big\langle g_i \otimes h_i, x \otimes a \big\rangle = \sum_{i=1}^{n} \big\langle g_i, x \big\rangle \big\langle h_i, a \big\rangle = \Big\langle \sum_{i=1}^{n} \big\langle h_i, a \big\rangle g_i, x \Big\rangle. \end{aligned}$$

This shows that

$$a.f = \sum_{i=1}^{n} \left\langle h_i, a \right\rangle g_i$$

and thus $A.f$ is contained in the linear span of $g_1, \ldots, g_n$, and in particular $A.f$ is finite dimensional.

Suppose then that $\dim (A.f) < \infty$. Let $(g_i)_{i=1}^r$ be a basis of $A.f$, and observe that writing $a.f$ in this basis we get $a.f = \sum_{i=1}^r \left\langle h_i, a \right\rangle g_i$ for some $h_i \in A^*$, $i = 1, 2, \ldots, r$. We can then compute for any $x, y \in A$

$$\left\langle \mu^*(f), x \otimes y \right\rangle = \left\langle f, xy \right\rangle = \left\langle y.f, x \right\rangle = \sum_i \left\langle h_i, y \right\rangle \left\langle g_i, x \right\rangle$$

to conclude that $\mu^*(f) = \sum_{i=1}^r g_i \otimes h_i \in A^* \otimes A^*$. □

It follows from the proof that for $f \in A^\circ$, the rank of $\mu^*(f) \subset A^* \otimes A^*$ is equal to the dimension of $A.f$. We in fact easily see that when $\mu^*(f) = \sum_{i=1}^r g_i \otimes h_i \in A^* \otimes A^*$ with r minimal, then $(g_i)_{i=1}^r$ is a basis of $A.f$ and $(h_i)_{i=1}^r$ is a basis of $f.A$.

Corollary 12. *If $f \in A^\circ$, then we have $\mu^*(f) \subset (A.f) \otimes (f.A) \subset A^\circ \otimes A^\circ$ and therefore*

$$\mu^*(A^\circ) \subset A^\circ \otimes A^\circ.$$

Proof. In the above proof we've written $\mu^*(f) = \sum_i g_i \otimes h_i$ with $g_i \in A.f$ and $h_i \in f.A$, so the first inclusion follows. But we clearly have also $A.f \subset A^\circ$ since for any $a \in A$ the submodule of A^* generated by the element $a.f$ is contained in $A.f$, and is therefore also finite dimensional. Similarly one gets $f.A \subset A^\circ$. □

We observe the following.

Corollary 13. *The restricted dual A° is spanned by the representative forms of finite dimensional A-modules.*

Proof. In Example 15 we have seen that the representative forms are always in the restricted dual. We must now show that any $f \in A^\circ$ can be written as a linear combination of representative forms. To this end we consider the finite dimensional submodule $A.f$ of A^*. Let $(g_i)_{i=1}^n$ be a basis of $A.f$, and assume without loss of generality that $g_1 = f$ and $g_i = b_i.f$ with $b_i \in A$, $i = 1, 2, \ldots, n$.

As above we observe that there exists $(h_i)_{i=1}^n$ in A^* such that $a.f = \sum_{i=1}^n \left\langle h_i, a \right\rangle g_i$ for all $a \in A$. We compute

$$a.g_j = (a\, b_j).f = \sum_{i=1}^{n} \left\langle h_i, a\, b_j \right\rangle g_i = \sum_{i=1}^{n} \left\langle b_j.h_i, a \right\rangle g_i,$$

so that the representative forms of $A.f$ in the basis (g_i) are $\lambda_{i,j} = b_j.h_i$. In particular since $b_1 = 1_A$ we have $h_i = \lambda_{i,1}$. It therefore suffices to show that f can be written as a linear combination of the elements h_i. But this is evident, since the (right) submodule $f.A$ of A^* contains f and is spanned by (h_i). □

We may write the conclusion above even more concretely as

$$f = f.1_A = \sum_i \left\langle g_i, 1_A \right\rangle h_i = \sum_i \left\langle g_i, 1_A \right\rangle \lambda_{i,1}.$$

Proof of Theorem 9. From Corollary 12 we see that we can interpret the structural maps as maps between the correct spaces,

$$\Delta = \mu^*\big|_{A^\circ} : A^\circ \to A^\circ \otimes A^\circ \qquad \text{and} \qquad \epsilon = \eta^*\big|_{A^\circ} : A^\circ \to \mathbb{C}.$$

To prove counitality, take $f \in A^\circ$ and write as before $\Delta(f) = \mu^*(f) = \sum_i g_i \otimes h_i$, and compute for any $x \in A$

$$\langle (\epsilon \otimes \mathrm{id}_{A^\circ})(\Delta(f)), x \rangle \;=\; \sum_i \epsilon(g_i)\, \langle h_i, x\rangle \;=\; \sum_i \langle g_i, 1_A\rangle\, \langle h_i, x\rangle \;=\; \langle \mu^*(f), 1_A \otimes x\rangle \;=\; \langle f, 1_A x\rangle \;=\; \langle f, x\rangle,$$

which shows $(\epsilon \otimes \mathrm{id}_{A^\circ})(\Delta(f)) = f$, and a similar computation shows $(\mathrm{id}_{A^\circ} \otimes \epsilon)(\Delta(f)) = f$. Coassociativity of μ^* follows from taking the transpose of the associativity of μ once one notices that the transpose maps have the appropriate alternative expressions

$$(\mathrm{id}_A \otimes \mu)^*\big|_{A^*\otimes A^*} \;=\; \mathrm{id}_A^* \otimes \mu^* \;=\; \mathrm{id}_{A^*} \otimes \mu^* \qquad \text{and} \qquad (\mu \otimes \mathrm{id}_A)^*\big|_{A^*\otimes A^*} \;=\; \mu^* \otimes \mathrm{id}_A^* \;=\; \mu^* \otimes \mathrm{id}_{A^*}$$

on the subspaces where we need them. □

To handle restricted duals of Hopf algebras, we present yet a few lemmas which say that the structural maps take values in the appropriate subspaces.

Lemma 14. *Let $B = (B, \mu, \Delta, \eta, \epsilon)$ be a bialgebra. Then we have $\Delta^*(B^\circ \otimes B^\circ) \subset B^\circ$. Also we have $\mu^*(\epsilon^*(1)) = \epsilon^*(1) \otimes \epsilon^*(1)$ so that $\epsilon^*(1) \in B^\circ$.*

Proof. Suppose $f_1, f_2 \in B^\circ$, and write

$$\mu^*(f_k) \;=\; \sum_i g_i^{(k)} \otimes h_i^{(k)} \qquad \text{for } k = 1, 2.$$

To show that $\Delta^*(f_1 \otimes f_2) \in B^\circ$, by definition we need to show that $\mu^*(\Delta^*(f_1 \otimes f_2)) \in B^* \otimes B^*$. Let $a, b \in B$ and notice that the axiom (H4) saves the day in the following calculation:

$$\begin{aligned}
\langle \mu^*\big(\Delta^*(f_1 \otimes f_2)\big), a \otimes b\rangle \;&=\langle f_1 \otimes f_2, \Delta\big(\mu(a \otimes b)\big)\rangle \\
&\overset{\text{(H4)}}{=} \sum_{(a),(b)} \langle f_1 \otimes f_2, a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)}\rangle \\
&= \sum_{(a),(b)} \langle f_1, a_{(1)} b_{(1)}\rangle\, \langle f_2, a_{(2)} b_{(2)}\rangle \\
&= \sum_{(a),(b)} \sum_{i,j} \langle g_i^{(1)}, a_{(1)}\rangle\, \langle h_i^{(1)}, b_{(1)}\rangle\, \langle g_j^{(2)}, a_{(2)}\rangle\, \langle h_j^{(2)}, b_{(2)}\rangle \\
&= \sum_{i,j} \langle g_i^{(1)} \otimes g_j^{(2)}, \Delta(a)\rangle\, \langle h_i^{(1)} \otimes h_j^{(2)}, \Delta(b)\rangle \\
&= \sum_{i,j} \langle \Delta^*(g_i^{(1)} \otimes g_j^{(2)}) \otimes \Delta^*(h_i^{(1)} \otimes h_j^{(2)}), a \otimes b\rangle\,.
\end{aligned}$$

We conclude that

$$\mu^*\big(\Delta^*(f_1 \otimes f_2)\big) \;=\; \sum_{i,j} \underbrace{\Delta^*\big(g_i^{(1)} \otimes g_j^{(2)}\big)}_{\in B^*} \otimes \underbrace{\Delta^*\big(h_i^{(1)} \otimes h_j^{(2)}\big)}_{\in B^*} \;\in\; B^* \otimes B^*,$$

and since the images under $\Delta^*|_{B^\circ \otimes B^\circ}$ of simple tensors are in B°, the assertion about Δ^* follows. This computation also shows that axiom (H4) holds in the restricted dual.

To prove the assertion about ϵ^*, note first that with the usual identifications $\langle \epsilon^*(1), a\rangle = \langle 1, \epsilon(a)\rangle = \epsilon(a)$. Take $a, b \in B$ and compute

$$\langle \mu^*(\epsilon^*(1)), a \otimes b\rangle \;=\; \langle 1, \epsilon(ab)\rangle \;=\; \epsilon(a)\,\epsilon(b) \;=\; \langle \epsilon^*(1) \otimes \epsilon^*(1), a \otimes b\rangle.$$

In fact this also shows that axiom (H5) holds in the restricted dual. □

Lemma 15. *Let $H = (H, \mu, \Delta, \eta, \epsilon, \gamma)$ be a Hopf algebra. Then we have $\gamma^*(H^\circ) \subset H^\circ$.*

Proof. Let $f \in H^\circ$, and for $a, b \in H$ compute

$$\begin{aligned}\langle \mu^*(\gamma^*(f)), a \otimes b\rangle &= \langle f, \gamma(ab)\rangle \;=\; \langle f, \gamma(b)\gamma(a)\rangle \;=\; \langle \mu^*(f), \gamma(b)\otimes\gamma(a)\rangle \\ &= \sum_i \langle g_i, \gamma(b)\rangle\, \langle h_i, \gamma(a)\rangle \;=\; \sum_i \langle \gamma^*(g_i), b\rangle\, \langle \gamma^*(h_i), a\rangle \;=\; \sum_i \langle \gamma^*(h_i)\otimes\gamma^*(g_i), a\otimes b\rangle.\end{aligned}$$

Thus we have

$$\mu^*(\gamma^*(f)) \;=\; \sum_i \gamma^*(h_i)\otimes\gamma^*(g_i) \;\in\; H^*\otimes H^*.$$

□

Sketch of a proof of Theorem 10. We have checked that the structural maps take values in the appropriate spaces (restricted dual or its tensor powers) when their domains of definition are restricted to the appropriate spaces. Taking transposes of all axioms of Hopf algebras, and noticing that the transposes of tensor product maps coincide with the tensor product maps of transposes on the subspaces of our interest, one can mechanically check all the axioms for the Hopf algebra H°. □