1 On diagonalization of matrices: the Jordan normal form

Here and in the rest of the course, vector spaces are over the field $\mathbb{C}$ of complex numbers unless otherwise stated.

Motivation and definition of generalized eigenvectors

Given a square matrix A, it is often convenient to diagonalize A. This means finding an invertible matrix P ("a change of basis"), such that the conjugated matrix $P A P^{-1}$ is diagonal. If, instead of matrices, we think of a linear operator A from vector space V to itself, the equivalent question is finding a basis for V consisting of eigenvectors of A.

Recall from basic linear algebra that (for example) any real symmetric matrix can be diagonalized. Unfortunately, this is not the case with all matrices.

Example 1. *Let $\lambda \in \mathbb{C}$ and*

$$A = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \in \mathbb{C}^{3\times 3}.$$

The characteristic polynomial of A is

$$p_A(x) = \det(x\mathbb{I} - A) = (x-\lambda)^3,$$

so we know that A has no other eigenvalues but λ. As usual, since $\det(A - \lambda\mathbb{I}) = 0$, the eigenspace pertaining to eigenvalue λ is nontrivial, $\dim(\mathrm{Ker}\,(A-\lambda\mathbb{I})) > 0$. Note that

$$A - \lambda\mathbb{I} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

so that the image of A is two dimensional, $\dim(\mathrm{Im}\,(A-\lambda\mathbb{I})) = 2$. By rank-nullity theorem,

$$\dim\Big(\mathrm{Im}\,(A-\lambda\mathbb{I})\Big) + \dim\Big(\mathrm{Ker}\,(A-\lambda\mathbb{I})\Big) = 3,$$

so the eigenspace pertaining to λ must be one-dimensional. Thus the maximal number of linearly independent eigenvectors of A we can have is one — in particular, there doesn't exist a basis of $\mathbb{C}^3$ consisting of eigenvectors of A.

We still take a look at the action of A in some basis. Let

$$w_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad w_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad w_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then the following "string" indicates how $A - \lambda\mathbb{I}$ maps these vectors

$$w_3 \overset{A-\lambda}{\mapsto} w_2 \overset{A-\lambda}{\mapsto} w_1 \overset{A-\lambda}{\mapsto} 0.$$

In particular we see that $(A-\lambda\mathbb{I})^3 = 0$.

The example illustrates the following definition.

Definition 1. *Let V be a vector space and $A : V \to B$ be a linear map. A vector $v \in V$ is said to be a <u>generalized eigenvector</u> of eigenvalue λ if for some positive integer p we have $(A-\lambda\mathbb{I})^p\, v = 0$. The set of these generalized eigenvectors is called the <u>generalized eigenspace</u> of A pertaining to eigenvalue λ.*

Clearly the case $p = 1$ corresponds to the usual eigenvectors.

The Jordan normal form

Although not every matrix has a basis of eigenvectors, we will see that every complex square matrix has a basis of generalized eigenvectors. More precisely, if V is a finite dimensional complex vector space and $A : V \to V$ is a linear map, then there exists eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ of A (not necessarily distinct) and a basis $\{w_m^{(j)} \; : \; 1 \leq j \leq k, 1 \leq m \leq n_j\}$ of V which consists of "strings" as follows

$$\begin{array}{ccccccccccc} w_{n_1}^{(1)} & \overset{A-\lambda_1}{\mapsto} & w_{n_1-1}^{(1)} & \overset{A-\lambda_1}{\mapsto} & \cdots & \overset{A-\lambda_1}{\mapsto} & w_2^{(1)} & \overset{A-\lambda_1}{\mapsto} & w_1^{(1)} & \overset{A-\lambda_1}{\mapsto} & 0 \\ w_{n_2}^{(2)} & \overset{A-\lambda_2}{\mapsto} & w_{n_2-1}^{(2)} & \overset{A-\lambda_2}{\mapsto} & \cdots & \overset{A-\lambda_2}{\mapsto} & w_2^{(2)} & \overset{A-\lambda_2}{\mapsto} & w_1^{(2)} & \overset{A-\lambda_2}{\mapsto} & 0 \\ & & & & & & \vdots & & \vdots & & \vdots \\ w_{n_k}^{(k)} & \overset{A-\lambda_k}{\mapsto} & w_{n_k-1}^{(k)} & \overset{A-\lambda_k}{\mapsto} & \cdots & \overset{A-\lambda_k}{\mapsto} & w_2^{(k)} & \overset{A-\lambda_k}{\mapsto} & w_1^{(k)} & \overset{A-\lambda_k}{\mapsto} & 0. \end{array} \tag{1.1}$$

Note that in this basis the matrix of A takes the "block diagonal form"

$$A = \begin{bmatrix} J_{\lambda_1;n_1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & J_{\lambda_2;n_2} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & J_{\lambda_3;n_3} & & \mathbf{0} \\ \vdots & \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & J_{\lambda_k;n_k} \end{bmatrix}, \tag{1.2}$$

where the blocks correspond to the subspaces spanned by $w_1^{(j)}, w_2^{(j)}, \ldots, w_{n_j}^{(j)}$ and they take the following form

$$J_{\lambda_j;n_j} = \begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_j & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_j & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_j \end{bmatrix} \in \mathbb{C}^{n_j \times n_j}.$$

Definition 2. *A matrix of the form (1.2) is said to be in* Jordan normal form *(or Jordan canonical form).*

The characteristic polynomial of the a matrix A in Jordan canonical form is

$$p_A(x) = \det(x\mathbb{I} - A) = \prod_{j=1}^{k} (x - \lambda_j)^{n_j}.$$

Note also that if we write a block $J_{\lambda;n} = \lambda\mathbb{I} + N$ as a sum of diagonal part $\lambda\mathbb{I}$ and upper triangular part N, then the latter is nilpotent: $N^n = 0$. In particular the assertion $p_A(A) = 0$ of the Cayley-Hamilton theorem follows immediately.

Below is the shortest and most concrete proof of the existence of Jordan normal form known to me.

Theorem 1. *Given any complex $n \times n$ matrix A, there exists an invertible matrix P such that the conjugated matrix $P A P^{-1}$ is in Jordan normal form.*

Proof. In view of the above discussion it is clear that the statement is equivalent to the following: if V is a finite dimensional complex vector space and $A : V \to V$ a linear map, then there exists a basis of V consisting of strings as in (1.1).

We prove the statement by induction on $n = \dim(V)$. The case $n = 1$ is clear. As an induction hypothesis, assume that the statement is true for all linear maps of vector spaces of dimension less than n.

Take any eigenvalue λ of A (any root of the characteristic polynomial). Note that

$$\dim\ (\mathrm{Ker}\,(A - \lambda \mathbb{I})) > 0,$$

and since $n = \dim\ (\mathrm{Ker}\,(A - \lambda \mathbb{I})) + \dim\ (\mathrm{Im}\,(A - \lambda \mathbb{I}))$, the dimension of the image of $A - \lambda \mathbb{I}$ is strictly less than n. Denote

$$R \ = \ \mathrm{Im}\,(A - \lambda \mathbb{I}) \qquad \text{and} \qquad r \ = \ \dim\,(R) \ < \ n.$$

Note that R is an invariant subspace for A, that is $A\,R \subset R$ (indeed, $A\,(A - \lambda \mathbb{I})\,v = (A - \lambda \mathbb{I})\,A\,v$). We can use the induction hypothesis to the restriction of A to R, to find a basis

$$\{w_m^{(j)} \ : \ 1 \leq j \leq k, 1 \leq m \leq n_j\}$$

of R in which the action of A is described by the strings as in (1.1).

Let $q = \dim\ (R \cap \mathrm{Ker}\,(A - \lambda \mathbb{I}))$. This means that in R there are q linearly independent eigenvectors of A with eigenvalue λ. The vectors at the right end of the strings span the eigenspaces of A in R, so we assume without loss of generality that the last q strings correspond to eigenvalue λ and others to different eigenvalues: $\lambda_1, \lambda_2, \ldots, \lambda_{k-q} \neq \lambda$ and $\lambda_{k-q+1} = \lambda_{k-q+2} = \cdots = \lambda_k = \lambda$. For all $k - q < j \leq k$ the vector $w_{n_j}^{(j)}$ is in R, so we can choose $y^{(j)} \in V$ such that $(A - \lambda \mathbb{I})\,y^{(j)} = w_{n_j}^{(j)}$. The vectors $y^{(j)}$ extend the last q strings from the left.

Find vectors $z^{(1)}, z^{(2)}, \ldots, z^{(n-r-q)}$ which complete the linearly independent collection $w_1^{(k-q+1)}, \ldots, w_1^{(k-1)}, w_1^{(k)}$ to a basis of $\mathrm{Ker}\,(A - \lambda \mathbb{I})$. We have now found n vectors in V, which form strings as follows

$$\begin{array}{ccccccccc}
 & & & & & & z^{(1)} & \overset{A-\lambda}{\mapsto} & 0 \\
 & & & & & & \vdots & & \vdots \\
 & & & & & & z^{(n-r-q)} & \overset{A-\lambda}{\mapsto} & 0 \\
 & & w_{n_1}^{(1)} & \overset{A-\lambda_1}{\mapsto} & \cdots & \overset{A-\lambda_1}{\mapsto} & w_1^{(1)} & \overset{A-\lambda_1}{\mapsto} & 0 \\
 & & \vdots & & & & \vdots & & \vdots \\
 & & w_{n_{k-q}}^{(k-q)} & \overset{A-\lambda_{k-q}}{\mapsto} & \cdots & \overset{A-\lambda_{k-q}}{\mapsto} & w_1^{(k-q)} & \overset{A-\lambda_{k-q}}{\mapsto} & 0 \\
y^{(k-q+1)} & \overset{A-\lambda}{\mapsto} & w_{n_{k-q+1}}^{(k-q+1)} & \overset{A-\lambda}{\mapsto} & \cdots & \overset{A-\lambda}{\mapsto} & w_1^{(k-q+1)} & \overset{A-\lambda}{\mapsto} & 0 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
y^{(k)} & \overset{A-\lambda}{\mapsto} & w_{n_k-1}^{(k)} & \overset{A-\lambda}{\mapsto} & \cdots & \overset{A-\lambda}{\mapsto} & w_1^{(k)} & \overset{A-\lambda}{\mapsto} & 0
\end{array} .$$

It suffices to show that these vectors are linearly independent. Suppose that a linear combination of them vanishes

$$\sum_{j=k-q+1}^{k} \alpha_j\, y^{(j)} + \sum_{j,m} \beta_{j,m}\, w_m^{(j)} + \sum_{l=1}^{n-r-q} \gamma_l\, z^{(l)} \ = \ 0.$$

From the string diagram we see that the image of this linear combination under $A - \lambda \mathbb{I}$ is a linear combination of the vectors $w_m^{(j)}$, which are linearly independent, and since the coefficient of $w_{n_j}^{(j)}$ is α_j, we get $\alpha_j = 0$ for all j. Now recalling that $\{w_m^{(j)}\}$ is a basis of R, and $\{w_1^{(j)} : k - q < j \leq k\} \cup \{z^{(l)}\}$ is a basis of $\mathrm{Ker}\,(A - \lambda \mathbb{I})$, and $\{w_1^{(j)} : k - q < j \leq k\}$ is a basis of $R \cap \mathrm{Ker}\,(A - \lambda \mathbb{I})$, we see that all the coefficients in the linear combination must vanish. This finishes the proof. □

Diagonalizable matrices can be thought of as a simple example of completely reducible representations: the vector space V is a direct sum of eigenspaces of the matrix. The underlying algebra that is represented in V is the quotient of the polynomial algebra by the ideal generated by the minimal polynomial of the matrix. In particular, if all the roots of the minimal polynomial have multiplicity one, then all representations are completely reducible. Non-diagonalizable matrices are a simple example of a failure of complete reducibility. The Jordan blocks $J_{\lambda_j;n_j}$ correspond to invariant subspaces, which are indecomposable, but not irreducible if $n_j > 1$.