Hopf algebras and representations — Spring 2011

Problem sheet 9 — Model solutions

Ex 1

Let $u \in A$ be such that $\wp(\wp(x)) = u x u^{-1}$ for all $x \in A$.

$$\wp(u) u x = \wp(u) \wp(\wp(x)) u = \wp(\wp(x) u) u$$
$$= \wp(u \wp^{-1}(x)) u = x \wp(u) u$$

$$\wp(u) u = u \wp^{-1}(u) \underset{\uparrow}{=} u \wp(u)$$
$$\wp(\wp(u)) = u$$

Ex 2

We have already shown that for any $m \in \mathbb{N}$, $n \in \mathbb{Z}$

$$\wp(b^m a^n) = (-1)^m q^{-\frac{m(m+1)}{2} - nm} b^m a^{-n-m}$$

By writing $k = m$, $l = -n-m \iff m = k$, $n = -k-l$ we note that $(\wp(b^m a^n))_{m \in \mathbb{N}, n \in \mathbb{Z}}$ are linearly ind. and span H_q. Therefore $\wp^{-1}$ exist and is given by

$$\wp^{-1}(b^k a^l) = (-1)^k q^{\frac{k(k+1)}{2} + k\cdot(-k-l)} b^k a^{-k-l}$$

$$= (-1) q^{-\frac{k(k-1)}{2} - kl} b^k a^{-k-l}$$

Ex 3

$$(\Delta_{\mathcal{D}} \otimes \mathrm{id}_{A \otimes A^*})(R) = \sum_{i=1}^{d} \Delta_{\mathcal{D}}(\overbrace{e_i \otimes 1^*}^{= \iota_A(e_i)}) \otimes (1 \otimes \delta^i)$$

$$= \sum_{i=1}^{d} \sum_{(e_i)} ((e_i)_{(1)} \otimes 1^*) \otimes ((e_i)_{(2)} \otimes 1^*) \otimes (1 \otimes \delta^i)$$

$$R_{13} R_{23} = \sum_{i,j} (e_i \otimes 1^*) \otimes (e_j \otimes 1^*) \otimes \underbrace{((1 \otimes \delta^i)(1 \otimes \delta^j))}_{= 1 \otimes (\delta^i \delta^j)}$$

Now let's evaluate both expressions at point $a \otimes b \otimes c$.
The first one is

$$\sum_{i=1}^{d} \sum_{(e_i)} \langle 1^*, a \rangle \langle 1^*, b \rangle \langle \delta^i, c \rangle (e_i)_{(1)} \otimes (e_i)_{(2)} \otimes 1$$

$$= \langle 1^*, a \rangle \langle 1^*, b \rangle \left[\sum_{i=1}^{d} \langle \delta^i, c \rangle \Delta(e_i) \right] \otimes 1$$

$$= \langle 1^*, a \rangle \langle 1^*, b \rangle (\Delta(c) \otimes 1).$$

The second is

$$\sum_{i,j} \langle 1^*, a \rangle \langle 1^*, b \rangle \overbrace{\langle \delta^i \delta^j, c \rangle}^{= \sum_{(c)} \langle \delta^i, c_{(1)} \rangle \langle \delta^j, c_{(2)} \rangle} e_i \otimes e_j \otimes 1$$

$$= \langle 1^*, a \rangle \langle 1^*, b \rangle \left[\sum_{(c)} \underbrace{\sum_{i,j} \langle \delta^i, c_{(1)} \rangle \langle \delta^j, c_{(2)} \rangle e_i \otimes e_j}_{= c_{(1)} \otimes c_{(2)}} \right] \otimes 1$$

$$= \langle 1^*, a \rangle \langle 1^*, b \rangle (\Delta(c) \otimes 1)$$

Ex4

(a) Product and unit in A^* are Δ^* and ε^* and can be written as

$$\langle \Delta^*(f \otimes f'), e_h \rangle = \langle f \otimes f', e_h \otimes e_h \rangle = \langle f, e_h \rangle \langle f', e_h \rangle$$

$$\langle \varepsilon^*(1), e_h \rangle = \varepsilon(e_h) = 1$$

for any $f, f' \in A^*$ and $h \in G$. For any $f \in A^*$, define $\phi_f : G \to \mathbb{C}$ by $\phi_f(g) = \langle f, e_g \rangle$. The mapping preserves product and unit:

$$\phi_{ff'}(g) = \langle ff', e_g \rangle = \langle f, e_g \rangle \langle f', e_g \rangle = \phi_f(g)\phi_{f'}(g)$$

$$\phi_{1^*}(g) = \langle \varepsilon^*(1), e_g \rangle = 1$$

where $1^* = \varepsilon^*(1)$. Thus $f \mapsto \phi_f$ is a homomorphism of algebras. Since the spaces have the same dimension, we need only check that the linear map $f \mapsto \phi_f$ is an injection. Suppose that

$$\phi_f(g) = 0 \quad \forall g \iff \langle f, g \rangle = 0 \ \forall g \iff f = 0$$

(b) Let $g, h, h' \in G$

$$\langle \mu^*(f_g), e_h \otimes e_{h'} \rangle = \langle f_g, e_{hh'} \rangle = \delta_{g, hh'}$$

$$= \langle \sum_{g' \in G} f_{g'} \otimes f_{g'^{-1}g}, e_h \otimes e_{h'} \rangle$$

$$\Rightarrow \mu^*(f_g) = \sum_{g' \in G} f_{g'} \otimes f_{g'^{-1}g}$$

$$\langle \eta^*(f_g), 1 \rangle = \langle f_g, e_{1_G} \rangle = \delta_{g, 1_G}$$

$$\langle \nu^*(f_g), e_h \rangle = \langle f_g, e_{h^{-1}} \rangle = \delta_{g, h^{-1}} = \langle f_{g^{-1}}, e_h \rangle$$

$$\Rightarrow \nu^*(f_g) = f_{g^{-1}}$$

Here $1_G \in G$ is the identity element of G.

(c) Let $g, h \in G$

$$\Delta_D(e_h \otimes f_g) = \sum_{(e_h)(f_g)} \left((e_h)_{(1)} \otimes (f_g)_{(2)}\right) \otimes \left((e_h)_{(2)} \otimes (f_g)_{(1)}\right)$$

$$= \sum_{g' \in G} (e_h \otimes f_{g'^{-1}g}) \otimes (e_h \otimes f_{g'})$$

$$\varepsilon_D(e_h \otimes f_g) = \varepsilon(e_h) \langle f_g, e_{1_G} \rangle = \delta_{g, 1_G}$$

$$\eta_D(1) = 1 \otimes 1^* = \sum_{g'} e_{1_G} \otimes f_{g'}$$

(d) Let $g, g', h, h' \in G$.

$$\mu_D\left((e_h \otimes f_g) \otimes (e_{h'} \otimes f_{g'})\right)$$

$$= \sum_{(f_g),(e_{h'})} \langle (f_g)_{(1)}, (e_{h'})_{(3)} \rangle \langle (f_g)_{(3)}, S^{-1}((e_{h'})_{(1)}) \rangle \cdot$$

$$\cdot \left((e_h (e_{h'})_{(2)}) \otimes ((f_g)_{(2)} f_{g'}) \right)$$

$$= \sum_{g_1, g_2 \in G} \langle f_{g_1}, e_{h'} \rangle \langle f_{g_2^{-1}g}, e_{(h')^{-1}} \rangle \; e_{hh'} \otimes f_{g_1^{-1}g_2} f_{g'}$$

where we have used $(\Delta \otimes id) \circ \Delta(e_{h'}) = e_{h'} \otimes e_{h'} \otimes e_{h'}$ and

$$(\mu^* \otimes id) \circ \mu^*(f_g) = \sum_{g_1, g_2 \in G} f_{g_1} \otimes f_{g_1^{-1}g_2} \otimes f_{g_2^{-1}g}$$

Now $\langle f_{g_1}, e_{h'} \rangle = \delta_{g_1, h'}$, $\langle f_{g_2^{-1}g}, e_{(h')^{-1}} \rangle = \delta_{g_2, gh'}$ and

$f_{g_1^{-1}g_2} f_{g'} = \delta_{g_1^{-1}g_2, g'} f_{g'}$.

$$\Rightarrow \mu_D\left((e_h \otimes f_g) \otimes (e_{h'} \otimes f_{g'})\right) = \delta_{g, h'g'(h')^{-1}} \; e_{hh'} \otimes f_{g'}$$

$$\mathcal{S}_{\mathcal{D}}(e_h \otimes f_g) = \sum_{(e_h),(f_g)} \langle (f_g)_{(1)}, \mathcal{S}^{-1}((e_h)_{(3)}) \rangle \langle (f_g)_{(3)}, (e_h)_{(1)} \rangle \cdot$$

$$\cdot\ \mathcal{S}((e_h)_{(2)}) \otimes (\mathcal{S}^*)^{-1}((f_g)_{(2)})$$

$$= \sum_{g_1, g_2 \in G} \underbrace{\langle f_{g_1}, e_{h^{-1}} \rangle}_{= \delta_{g_1, h^{-1}}} \underbrace{\langle f_{g_2^{-1} g}, e_h \rangle}_{= \delta_{g_2, g h^{-1}}} e_{h^{-1}} \otimes f_{g_2^{-1} g_1}$$

$$= e_{h^{-1}} \otimes f_{h g^{-1} h^{-1}}$$

where we used, among other things, that $\mathcal{S}^{-1} = \mathcal{S}$ and $(\mathcal{S}^*)^{-1} = \mathcal{S}^*$ in this example.