Hopf algebras and representations
Spring 2011

Problem sheet 2
Model solutions

Ex 1

If V is a vector space, then V has a basis (Hamel basis) $(e_x)_{x\in X}$ where X is some index set. So any element $v \in V$ can be written as

$$v = \sum_{x\in X} \alpha_x e_x$$

where α_x is non-zero for only finitely many $x \in X$. Therefore each V can be seen as a subspace of function space (more accurately isomorphic to a subspace of a function space).

(a) For $f \in \mathbb{K}^X$, $g \in \mathbb{K}^Y$, we set $(f\otimes g)(x,y) = f(x)g(y)$ and $V \otimes W = \operatorname{span}\{f\otimes g : f \in V, g \in W\}$.

Let $(f_i)_{i\in I} \subset V$ and $(g_j)_{j\in J} \subset W$ be linearly indep. collections. Let $\alpha_{ij} \in \mathbb{K}$ $\forall i \in I, \forall j \in J$ be such that

$$\sum_{\substack{i\in I\\ j\in J}} \alpha_{ij} f_i \otimes g_j = 0 \quad \text{and} \quad \alpha_{ij} \neq 0 \text{ for only finitely many } i \text{ and } j$$

$$\Rightarrow \quad 0 = \Big(\sum_{\substack{i\in I\\ j\in J}} \alpha_{ij} f_i \otimes g_j\Big)(x,y) = \sum_{\substack{i\in I\\ j\in J}} \alpha_{ij} f_i(x) g_j(y)$$

$$= \sum_{i\in I} \Big[\sum_{j\in J} \alpha_{ij} g_j(y)\Big] f_i(x) \qquad \forall x\in X, \forall y \in Y$$

If we consider this for fixed y as a function of x, then by independence of $(f_i)_{i\in I}$

$$\Rightarrow \quad \sum_{j\in J} \alpha_{ij} g_j(y) = 0 \qquad \forall i, \forall y$$

$$\Rightarrow \quad \alpha_{ij} = 0 \qquad \forall i, \forall j$$

Therefore $(f_i \otimes g_j)_{i\in I, j\in J}$ is linearly independent.

(b) $V = \mathrm{span}[(f_i)_{i\in I}]$, $W = \mathrm{span}[(g_j)_{j\in J}]$

If $h \in V \otimes W$, then exist $\tilde{f}_k \in V$ and $\tilde{g}_k \in W$ $k = 1, 2, \dots, n$ for some n such that

$$h = \sum_{k=1}^{n} \tilde{f}_k \otimes \tilde{g}_k$$

Now we can write $\tilde{f}_k = \sum_{i\in I} c_{ki} f_i$, $\tilde{g}_k = \sum_{j\in J} d_{kj} g_k$ where $c_{ki} \in \mathbb{K}$ and $d_{kj} \in \mathbb{K}$ are non-zero only for finitely many indices.

$$\Rightarrow h(x,y) = \sum_{k=1}^{n} \tilde{f}_k(x)\tilde{g}_k(y) = \sum_{k=1}^{n} \left[\sum_{i\in I} c_{ki} f_i(x)\right]\left[\sum_{j\in J} d_{kj} g_j(y)\right]$$

$$= \sum_{k=1}^{n} \sum_{i\in I} \sum_{j\in J} c_{ki} d_{kj} f_i(x) g_j(y)$$

$$= \sum_{\substack{i\in I \\ j\in J}} \left[\sum_{k=1}^{n} c_{ki} d_{kj}\right] f_i(x) g_j(y)$$

Therefore any $h \in V \otimes W$ can be written as (finite) linear combination of $(f_i \otimes g_j)_{i\in I, j\in J}$.

(c) Basis = linearly independent subset which spans the vector space

(a), (b) $\Rightarrow$ $(f_i \otimes g_j)$ is a basis if (f_i) and (g_j) are.

Let $\phi(f,g) = f \otimes g$ which is a bilinear map from $V \times W$ to $V \otimes W$. If $\beta : V \times W \to U$ is bilinear, then set $\beta_{ij} = \beta(f_i, g_j) \in U$ and define linear $\tilde{\beta} : V \otimes W \to U$ by $\tilde{\beta}(f_i \otimes g_j) = \beta_{ij}$ and by linear extension. Then if $f = \sum_i c_i f_i$, $g = \sum_j d_j g_j$

$$\beta(f,g) = \sum_{i,j} c_i d_j \beta(f_i, g_j) = \sum_{i,j} c_i d_j \tilde{\beta}(f_i \otimes g_j)$$

$$= \tilde{\beta}\left(\left(\sum_i c_i f_i\right) \otimes \left(\sum_j d_j g_j\right)\right) = \tilde{\beta}(\phi(f,g))$$

Ex 2

(a) Clearly $v \mapsto \varphi(v) w$ is a linear map from V to W and hence defines an element of $\mathrm{Hom}(V, W)$.

Since $(w, \varphi) \mapsto \varphi(\cdot) w$ is bilinear from $W \times V^*$ to $\mathrm{Hom}(V, W)$, it defines a linear map from $W \otimes V^*$ to $\mathrm{Hom}(V, W)$ by the universal property of tensor product.

If this map is not injective, then $\exists\, h \in W \otimes V^*$ $h = \sum_{i=1}^{r} w_i \otimes \varphi_i$, where $r = \mathrm{rank}(h)$, such that h is mapped on $0 \in \mathrm{Hom}(V, W)$.

$$\Rightarrow \quad 0 = \sum_{i=1}^{r} \varphi_i(v)\, w_i \quad \forall v \in V$$

By a lemma from the lectures, (w_i) is linearly independent. $\Rightarrow$ $\varphi_i(v)$ $\forall i$ $\forall v$ $\Rightarrow$ $h = 0$ $\lightning$

Hence the above map is injective.

(b) When V and W are finite dimensional, then the vector spaces $W \otimes V^*$ and $\mathrm{Hom}(V, W)$ are finite dimensional. Hence it is enough to note that dimensions of these spaces are equal. Namely, $\dim(W \otimes V^*) = \dim(W)\dim(V^*)$ $= \dim(W)\dim(V)$ by Ex 1 and by $V \cong V^*$ and $\dim(\mathrm{Hom}(V, W)) = \dim(V)\dim(W)$, which can assumed to be known from linear algebra. Hence the mapping in (a) is a bijection.

Let $h = \sum_{i=1}^{r} w_i \otimes \varphi_i$, where $r = \mathrm{rank}(h)$. Since $(\varphi_i)_{i=1}^{r}$ are linearly independent, we can choose a basis $(e_i)_{i=1}^{\dim(V)}$ for V so that $\varphi_i(e_j) = \delta_{i,j}$ for $1 \le i \le r$ and $1 \le j \le \dim(V)$. Then for any $v = \sum_i c_i e_i$,

$$\sum_{i=1}^{r} \varphi_i(v)\, w_i = \sum_{i=1}^{r} c_i w_i$$

Therefore if $A_h \in \mathrm{Hom}(V,W)$ is the image of h under the mapping of (a), then $\mathrm{Im}(A_h) = \mathrm{span}((w_i)_{i=1}^r)$.
Since (w_i) are lin. ind., $\mathrm{rank}(A_h) = \dim(\mathrm{Im}(A_h)) = r$.

c) Recall that $g.(v \otimes w) = (g.v) \otimes (g.\varphi)$ and $(g.\varphi)(v) = \varphi(g^{-1}.v)$
for any $g \in G$, $v \in V$, $w \in W$, $\varphi \in V^*$.

In the mapping of (a), $g.(w \otimes \varphi) = (g.w) \otimes (g.\varphi)$ is
mapped to $v \mapsto (g.\varphi)(v)\, g.w = \varphi(g^{-1}.v)\, g.w$
which agrees with the other dewinition of
representation on $\mathrm{Hom}(V,W)$: $g.T = \rho_2(g) \circ T \circ \rho_1(g^{-1})$
where $T \in \mathrm{Hom}(V,W)$ and $\rho_1 : G \to GL(V_1)$, $\rho_2 : G \to GL(V_2)$

Ex 3

(a) G abelian. Let $g \in G$. Since $gg' = g'g$ for any $g' \in G$, for any representation $\rho: G \to GL(V)$ $\rho(g)$ is a G-module map. By Schur's lemma, in irreducible representation $\rho(g) = \lambda(g)\,\mathrm{id}$ for some $\lambda(g) \in \mathbb{C}$.
Now also by irreducibility $\dim(V) = 1$.

(b) C_n cyclic group of order n $c^n = e$

C_n is abelian $\Rightarrow$ irreducible representations are one-dimensional, i.e. $\rho(g) \in \mathbb{C}$ and $\rho(g)^n = 1$.

Ex 4

If V, V', W, W' are vector spaces and $A: V \to W$ and $B: V' \to W'$ are linear mappings, then $(A \otimes B)(v \otimes w) \overset{d}{=} (Av) \otimes (Bw)$ defines a linear mapping $V \otimes V' \to W \otimes W'$. Suppose for simplicity that $V = V' = W = W' = \mathbb{C}^2$ and write A and B as matrices then in the basis $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$ the matrix

$$A \otimes B = \begin{pmatrix} A_{11} B & A_{12} B \\ A_{21} B & A_{22} B \end{pmatrix}$$

[Kronecker product of matrices]

Either by above or direct calculation using $g.(v \otimes v') = (g.v) \otimes (g.v')$

$$\rho(r) = \begin{bmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{bmatrix}, \quad \rho(m) = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}$$

Where $\rho : D_4 \to GL(V \otimes V)$ is the tensor representation and the matrices are written in the basis $(e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2)$. For example

$$r.(e_1 \otimes e_1) = (r.e_1) \otimes (r.e_1) = e_2 \otimes e_2$$

$$m.(e_1 \otimes e_1) = (m.e_1) \otimes (m.e_1) = (-e_1) \otimes (-e_1) = e_1 \otimes e_1$$

Let
$$\begin{cases} v_1 = e_1 \otimes e_1 + e_2 \otimes e_2 \\ v_2 = e_1 \otimes e_1 - e_2 \otimes e_2 \\ v_3 = e_1 \otimes e_2 + e_2 \otimes e_1 \\ v_4 = e_1 \otimes e_2 - e_2 \otimes e_1 \end{cases}$$

Then

$$\begin{cases} r.v_1 = v_1, \; m.v_1 = v_1 & \text{i.e. } r = 1, m = 1 \\ r.v_2 = -v_2, \; m.v_2 = v_2 & \text{i.e. } r = -1, m = 1 \\ r.v_3 = -v_3, \; m.v_3 = -v_3 & \text{i.e. } r = -1, m = -1 \\ r.v_4 = v_4, \; m.v_4 = -v_4 & \text{i.e. } r = 1, m = -1 \end{cases}$$