Hopf algebras and Representations Problem sheet 1
Spring 2011 Model solutions

Ex 1 (a) Let $A = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$

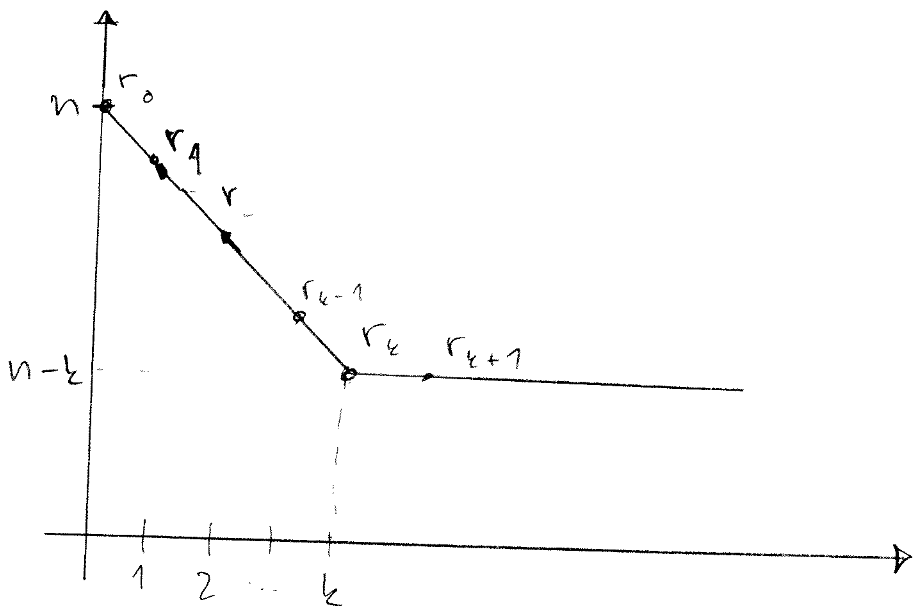
Both of them have characteristic polynomial equal to z^4 and minimal polynomial z^2.
They are not similar since $\operatorname{rank}(A) = 2 \neq 1 = \operatorname{rank}(B)$

(b) Similarity is equivalence relation. Therefore $C_1 = P C_2 P^{-1}$ for some P [here $P = P_1 P_2^{-1}$].

For any fixed λ so that $J_{\lambda;k}$ is Jordan block of C_1 for some k, define

$$r_j = \operatorname{rank}\left((C_1 - \lambda I)^j\right) = \operatorname{rank}\left((A - \lambda I)^j\right)$$
$$= \operatorname{rank}\left((C_2 - \lambda I)^j\right)$$

When there is only one such block, say, $J_{\lambda;k}$ then

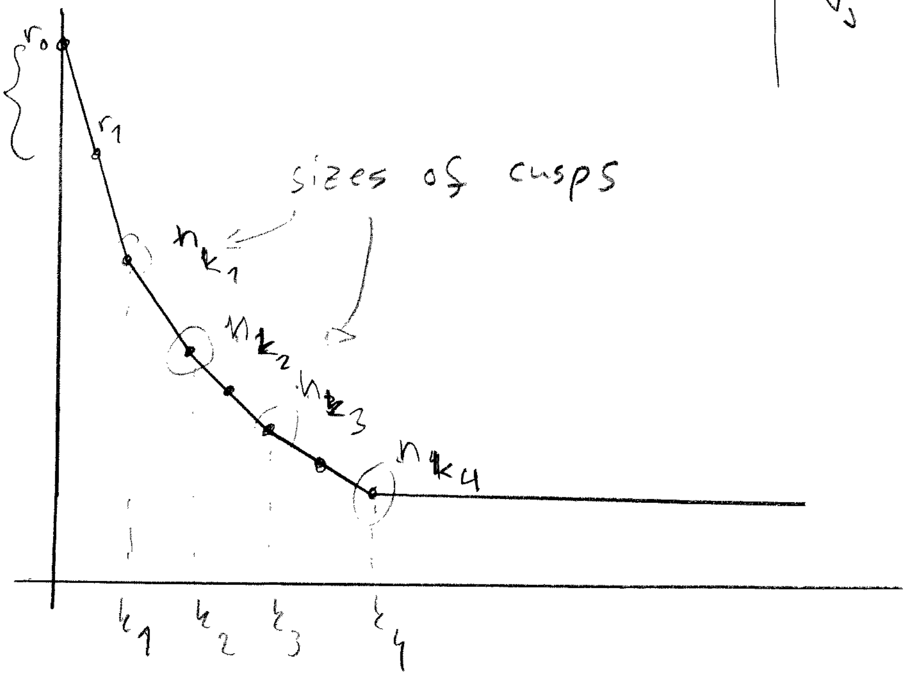


Size of the block $= k$ can be read from the picture.
It is the unique k so that $r_{k-1} + r_{k+1} - 2r_k = 1$.

More general

Note: for each J, there is an invariant subspace V_J and $V = \bigoplus_{J \text{ a Jordan block of } C_1} V_J$.

geom. mult. = # of Jordan blocks with λ



$\Rightarrow$ exactly n_{k_j} blocks of size k_j

Since the numbers (r_j) are invariant in similarity transforms, also $n_j = r_{j-1} + r_{j+1} - 2r_j$ are.

(c) Permutation matrices are of the form

$$\begin{pmatrix} & & 1 & \\ 1 & & & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{pmatrix}$$

column permutations of columns of I

Each permutation matrix is a product of transpositions, i.e. matrices of the form

$$P(i,j) = \begin{array}{c} \\ i \\ \\ j \end{array} \begin{pmatrix} 1 & & & & \\ & 0 & & 1 & \\ & & 1 & & \\ & 1 & & 0 & \\ & & & & 1 \end{pmatrix} = I - (e_i - e_j)(e_i - e_j)^T$$

Since $P(i,j)^2 = I$ it holds that $PP^T = I$ for any permutation matrix! Now

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & & \\ & & & & I_{k+1} \\ & & & I_k & \\ & & & & & \ddots 1 \end{pmatrix} \begin{pmatrix} \ddots & & & \\ & J_k & & \\ & & J_{k+1} & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & & \\ & & & & I_k \\ & & & I_{k+1} & \\ & & & & & \ddots \end{pmatrix} = \begin{pmatrix} \ddots & & & \\ & & J_{k+1} & \\ & J_k & & \\ & & & \ddots \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & & \\ & & & & I_k \\ & & & I_{k+1} & \\ & & & & & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} \ddots & & & \\ & J_{k+1} & & \\ & & J_k & \\ & & & \ddots \end{pmatrix}$$

Other permutations of the Jordan blocks can be constructed from these "flips"!

Ex 2

(a) For each $g \in G$ (when G is finite group), there is n so that $g^n = e$. Smallest such n is "the order of g" $= |\underbrace{<g>}_{\text{cyclic subgroup gen. by } g}|$ and $|<g>|$ divides $|G|$. $\Rightarrow g^{|G|} = e$

If v is eigenvector of $\rho(g)$ with eigenvalue λ then

$$v = \rho(e)v = \rho(g^{|G|})v = \rho(g)^{|G|}v = \lambda^{|G|}v$$

$$\Rightarrow \lambda^{|G|} = 1$$

(b) Let $g \in G$. Then $\rho'(g).f \in V^*$ $\forall f$ since

$$v \longmapsto <f, \rho(g^{-1}).v>$$

is linear map $V \to \mathbb{C}$. Furthermore $f \longmapsto \rho'(g).f$ is linear:

$$<\rho'(g).(\alpha_1 f_1 + \alpha_2 f_2), v> = <\alpha_1 f_1 + \alpha_2 f_2, \rho(g^{-1}).v>$$

$$= \alpha_1 <f_1, \rho(g^{-1}).v> + \alpha_2 <f_2, \rho(g^{-1}).v>$$

$$= \alpha_1 <\rho'(g).f_1, v> + \alpha_2 <\rho'(g).f_2, v>$$

Hence $\rho'(g)$ is a linear mapping $V^* \to V^*$.

$\rho'(g)$ is invertible if $[<\rho'(g).f, v> = 0 \ \forall v \Rightarrow f = 0]$ which holds since if $0 = <\rho'(g).f, v> = <f, \rho(g^{-1}).v>$ $\forall v$ then $f = 0$.

ρ' is a representation since $\rho'(g\tilde{g}) = \rho'(g)\rho'(\tilde{g})$:

$$<\rho'(g\tilde{g}).f, v> = <f, \rho(\tilde{g}^{-1}g^{-1}).v>$$

$$= <f, \rho(\tilde{g}^{-1})(\rho(g^{-1}).v)> = <\rho'(\tilde{g}).f, \rho(g^{-1}).v>$$

$$= <\rho'(g^{-1}).(\rho'(\tilde{g}).f), v>$$

(c) By a lemma from the lectures, we can assume that $\rho(g)$ is diagonalizable for all g.

For fixed g, take (v_i) an eigenbasis for $\rho(g)$ i.e. $\exists\,(\lambda_i)$ s.t.

$$\rho(g)\,v_i = \lambda_i v_i$$

Take a dual basis (f_i) for V^*, i.e.

$$\langle f_i, v_j\rangle = \delta_{ij}$$

Then the trace of $\rho(g)$ is

$$\mathrm{Tr}(\rho(g)) = \sum_{i=1}^{n} \langle f_i, \rho(g) v_i\rangle = \sum_{i=1}^{n} \lambda_i$$

Note also that $\rho(g^{-1})v_i = \rho(g)^{-1} v_i = \lambda_i^{-1} v_i$.

Now since $V^{**} = V$, the trace of $\rho'(g)$ is

$$\mathrm{Tr}(\rho'(g)) = \sum_{i=1}^{n} \langle \rho'(g).f_i, v_i\rangle = \sum_{i=1}^{n} \langle f_i, \rho(g^{-1}).v_i\rangle$$

$$= \sum_{i=1}^{n} \lambda_i^{-1} = \overline{\mathrm{Tr}(\rho(g))}$$

since by (a) $\lambda_i^{-1} = \overline{\lambda_i}$.

Ex 3

Let $\sigma = (123)$, $\tau = (12)$. The elements of S_3 can be written in terms of these as $\{e, \tau, \tau\sigma, \sigma\tau, \sigma, \sigma^2\}$ and therefore it is enough to write the representation only for σ and τ. [σ and τ generate S_3, if you wish to say like that]. The properties we will need in the following are $\sigma^3 = e$, $\tau^2 = e$ and $\underbrace{\tau\sigma\tau = \sigma^2}_{(*)}$.

Let $\rho : S_3 \to GL(V)$ be a two dimensional irreducible representation. Then $\rho(\sigma)v = \lambda v$ for some $0 \neq v \in V$ and $\lambda \in \mathbb{C}$. Note that $\lambda^3 = 1$. Now

$$\rho(\sigma)\,[\rho(\tau)v] \overset{(*)}{=} \rho(\tau)\rho(\sigma)^2 v = \lambda^2 \rho(\tau)v$$

So also $\rho(\tau)v$ is an eigenvector of $\rho(\sigma)$.
Since V is irreducible $\rho(\tau)v$ and v are lin. independent. [Note otherwise necessarily also $\lambda = 1$.]
In the basis $\{v, \rho(\tau)v\}$

$$\rho(\sigma) = \begin{bmatrix} \lambda & \\ & \lambda^2 \end{bmatrix}, \quad \rho(\tau) = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$$

Note that if $\lambda = 1$ this is not irreducible ($sp(\{\begin{bmatrix}1\\1\end{bmatrix}\})$ is invariant subspace).
Hence $\lambda = e^{\pm i \frac{2\pi}{3}}$.

Those two repentations are isomorphic.

Ex 4

(a) General elements of the form r^k or mr^k
$k = 0,1,2,3$ (total of 8 elements). By the
last relation $rmrm = e \iff rm = mr^{-1} = mr^3$
$\iff mr = r^{-1}m$

$$\begin{cases} r^j r^k r^{-j} = r^k \\ mr^j r^k r^{-j} m = mr^k m = m^2 r^{-k} = r^{-k} \end{cases}$$

$\Rightarrow \{e\}, \{r, r^3\}, \{r^2\}$ are conj. classes

$$\begin{cases} r^j mr^k r^{-j} = mr^{k-2j} \\ mr^j mr^k r^{-j} m = mr^{2j-k} \end{cases}$$

$\Rightarrow \{m, mr^2\}, \{mr, mr^3\}$ are conj. classes

(b) If $\rho: D_4 \to \mathbb{C}$ is representation, the relations can be written as

$$\rho(r)^4 = 1, \quad \rho(m)^2 = 1, \quad \rho(r)^2\rho(m)^2 = 1$$

$$\iff \rho(r)^2 = 1, \quad \rho(m)^2 = 1$$

Hence $\rho(r) = \pm 1$, $\rho(m) = \pm 1$

(c) Note that $\rho(r) = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$, $\rho(m) = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}$ satisfy the relations, therefore such homomorphism exists.

If it is not irreducible, then exists one-dimensional invariant subspace $sp(\{v\})$. Hence $\exists \lambda, \mu \in \mathbb{C}$ s.t. $\rho(r)v = \lambda v$, $\rho(m)v = \mu v$, but the matrices don't have any common eigenvectors. $\lightning$