Hopf algebras and representations Spring 2011

Problem sheet 13 Model solutions

Ex 1

(a) Write $(R_{21}R)^{-1} = \sum_{i=1}^{m} a_i \otimes b_i$. Since $\varepsilon(\theta) = 1$, by (H3)

$$1 = \mu \circ (\mathcal{S} \otimes id) \circ \Delta(\theta) = \mu \circ (\mathcal{S} \otimes id)\left(\sum_{i=1}^{m} (a_i\theta) \otimes (b_i\theta)\right)$$

$$= \sum_{i=1}^{m} \mathcal{S}(a_i\theta) b_i \theta = \theta \left(\sum_{i=1}^{m} \mathcal{S}(a_i) b_i\right)\theta$$

Hence θ has both left and right inverse, θ^{-1} exists and $\theta^{-1} = \sum_{i=1}^{m} \mathcal{S}(a_i) b_i \theta = \sum_{i=1}^{m} \theta \mathcal{S}(a_i) b_i$.

(b) If $f : V \to W$ is A-module map, then for any $v \in V$

$$\Theta_W \circ f(v) = \theta^{-1}. f(v) = f(\theta^{-1}. v) = f \circ \Theta_V(v)$$

Hence $\Theta_W \circ f = f \circ \Theta_V$.

Next note that $\mathcal{S}(\theta^{-1}) = \mathcal{S}(\theta)^{-1} = \theta^{-1}$. For any $\varphi \in V^*$ and $v \in V$

$$\langle \Theta_{V^*}(\varphi), v \rangle = \langle \theta^{-1}. \varphi, v \rangle = \langle \varphi, \mathcal{S}(\theta^{-1}).v \rangle$$

$$= \langle \varphi, \theta^{-1}.v \rangle = \langle \varphi, \Theta_V(v) \rangle = \langle (\Theta_V)^*(\varphi), v \rangle$$

Hence $\Theta_{V^*} = (\Theta_V)^*$.

Next note that $\Delta(\theta^{-1}) = \Delta(\theta)^{-1} = (\theta^{-1} \otimes \theta^{-1}) R_{21} R$.

Now

$$c_{W,V} \circ c_{V,W} = S_{W,V} \circ ((\rho_W \otimes \rho_V)(R)) \circ S_{V,W} \circ$$

$$\circ ((\rho_V \otimes \rho_W)(R)) = ((\rho_V \otimes \rho_W)(R_{21})) \circ ((\rho_V \otimes \rho_W)(R))$$

$$\Rightarrow \rho_{V \otimes W}(\theta^{-1}) = (\rho_V \otimes \rho_W)(\Delta(\theta^{-1})) = (\Theta_V \otimes \Theta_W) \circ c_{W,V} \circ c_{V,W}$$

Ex 2

(a) Use the canonical identification $\mathbb{C} \cong \mathbb{C} \otimes \mathbb{C}$ in a suitable way to see that

$$\varepsilon(u) = (\varepsilon \otimes \varepsilon) \circ (S \otimes id_A)(R_{21})$$

$$= ((\varepsilon \circ S) \otimes id_{\mathbb{C}}) \underbrace{(id_A \otimes \varepsilon)(R_{21})}_{=1}$$

$$= 1$$

(b)

$$R_{21} R \Delta(x) = R_{21} \Delta^{cop}(x) R$$

$$= S_{A,A}(R\Delta(x))R = S_{A,A}(\Delta^{cop}(x)R)R$$

$$= \Delta(x) R_{21} R$$

(c) Now write $R = \sum_{i=1}^{n} s_i \otimes t_i$ and therefore $u = \sum_{i=1}^{n} S(t_i) s_i$. Let's list a couple of auxiliary formulas based on properties of R:

$$\sum_{i=1}^{n} \Delta(s_i) \otimes t_i = (\Delta \otimes id)(R) = R_{13} R_{23}$$

$$= \sum_{i,j=1}^{n} s_i \otimes s_j \otimes (t_i t_j) \qquad (1)$$

$$\sum_{i=1}^{n} s_i \otimes \Delta(t_i) = (id \otimes \Delta)(R) = R_{13} R_{12}$$

$$= \sum_{i,j=1}^{n} s_j s_i \otimes t_i \otimes t_j \qquad (2)$$

$$\sum_{i,j,k=1}^{n} s_i s_j \otimes S(t_i) S(s_k) \otimes S(t_j) t_k$$

$$= (id \otimes S \otimes S)\Big(\sum_{i,j,k=1}^{n} s_i s_j \otimes s_k t_i \otimes S^{-1}(t_k) t_j$$

$$= (id \otimes S \otimes S)\big(R_{23}^{-1} R_{12} R_{13} \big) = (id \otimes S \otimes S)\big(R_{13} R_{12} R_{23}^{-1}\big)$$

$$= (\mathrm{id} \otimes S \otimes S)\left(\sum_{i,j,k=1}^{n} s_j s_i \otimes t_i s_k \otimes t_j S^{-1}(t_k) \right)$$

$$= \sum_{i,j,k=1}^{n} s_j s_i \otimes S(s_k) S(t_i) \otimes t_k S(t_j) \qquad (3)$$

Now

$$\Delta(u) = \sum_{i=1}^{n} \Delta(S(t_i))\Delta(s_i) = \sum_{i=1}^{n} \left((S \otimes S) \circ \Delta^{cop}(t_i)\right)\Delta(s_i)$$

$$\overset{(2)}{=} \sum_{i,j=1}^{n} \left(S(t_i) \otimes S(t_j) \right) \quad \Delta(s_i)\Delta(s_j)))$$

$$\overset{(1)}{=} \sum_{i,j,k,l=1}^{n} \left(S(t_i t_j) \otimes S(t_k t_l) \right) (s_i \otimes s_j)(s_k \otimes s_l)$$

$$= \sum_{i,j,k,l=1}^{n} \left(S(t_j) S(t_i) s_i s_k \right) \otimes \left(S(t_l) S(t_k) s_j s_l \right)$$

$$= \sum_{j,k,l=1}^{n} \left(S(t_j)\, u\, s_k \right) \otimes \left(S(t_l) S(t_k) s_j s_l \right)$$

$$= \sum_{j,k,l=1}^{n} \left(S(t_j) S(S(s_k))\, u \right) \otimes \left(S(t_l) S(t_k) s_j s_l \right)$$

$$\overset{(S \otimes S)(R) = R}{=} \sum_{j,k,l=1}^{n} \left(S(t_j) S(s_k)\, u \right) \otimes \left(S(t_l)\, t_k s_j s_l \right)$$

$$\overset{(3)}{=} \sum_{j,k,l=1}^{n} \left(S(s_k) S(t_j)\, u \right) \otimes \left(t_k S(t_l) s_j s_l \right)$$

$$= R^{-1} (1 \otimes u) \left((S \otimes \mathrm{id})(R_{21}) \right) (u \otimes 1)$$

$$= R^{-1} \left((S \otimes (S \circ S))(R_{21}) \right) (u \otimes u)$$

$$= R^{-1} \left((\mathrm{id} \otimes S)(R_{21}) \right) (u \otimes u) = R^{-1} R_{21}^{-1} (u \otimes u)$$

(d) Note that

$$\Delta(u^{-1}) = \Delta(u)^{-1} = (u^{-1} \otimes u^{-1}) R_{21} R$$

$$\Delta(\wp(u^{-1})) = (\wp \otimes \wp) \circ \Delta^{cop}(u^{-1}) = (\wp \otimes \wp)((u^{-1} \otimes u^{-1}) R R_{21})$$

$$= R_{21} R (\wp(u^{-1}) \otimes \wp(u^{-1}))$$

Therefore

$$\Delta(u \wp(u^{-1})) = \Delta(u) \quad R_{21} R (\wp(u^{-1}) \otimes \wp(u^{-1}))$$

$$= R_{21} R \; \Delta(u) (\wp(u^{-1}) \otimes \wp(u^{-1}))$$

$$= (u \wp(u^{-1})) \otimes (u \wp(u^{-1})).$$

Ex 3

(a) Observe first that

$$K E^a F^b K^c K^{-1} = q^{2a-2b} E^a F^b K^c$$

$$\Rightarrow \{ x \in A : K x K^{-1} = x \} = \mathrm{span}\left(E^a F^a K^c\right)_{(a,c)\in\{0,1,\dots,e-1\}^2}$$

Now

$$E\, E^a F^a K^c - E^a F^a K^c E$$

$$= E^a \left(E F^a - q^{2c} F^a E \right) K^c$$

$$= E^a \left((1-q^{2c}) E F^a + q^{2c} \frac{[a]_q}{q-q^{-1}} F^{a-1} \left(q^{1-a} K - q^{a-1} K^{-1}\right)\right) K^c$$

$$= p_{a,c} E^{a+1} F^a K^c + r_{a,c} E^a F^{a-1} K^{c+1} + s_{a,c} E^a F^{a-1} K^{c-1}$$

for some $p_{a,c}, r_{a,c}, s_{a,c} \in \mathbb{C}$. This formula makes sense for $a=0$, if we set $r_{0,c} = s_{0,c} = 0$. Now an element $x = \sum_{a,c} \alpha_{a,c} E^a F^a K^c$, $\alpha_{a,c} \in \mathbb{C}$, commutes with E if the coefficients satisfy the equations

$$p_{a,c}\alpha_{a,c} + r_{a+1,c-1}\alpha_{a+1,c-1} + s_{a+1,c+1}\alpha_{a+1,c+1} = 0$$

for all $0 \le a \le e-2$, $0 \le c \le e-1$. It is easy to see that these equations linearly independent: each equation contains only one term of the "order" a and others are higher order $a+1$. Hence any linear combination of the equation which vanishes, has to have order $a=0$ coefficients equal to zero, then $a=1$ etc.

To see that the elements $1, C, C^2, \dots, C^{e-1}$ are linearly independent, write

$$C^n = \sum_{k=0}^{n} \binom{n}{k} (q-q^{-1})^{-2k} (EF)^{n-k} (q^{-1}K + qK)$$

$$= E^nF^n + A_n$$

where $A_n \in \operatorname{span}(E^aF^aK^c)_{0\le a\le n-1,\, 0\le c\le e-1}$.

This fact follows straight from the relations of A.

Now since E^nF^n, $0\le n\le e-1$, are lin. ind. so are C^n.

Since the dimension of the subalgebra commuting with K and E is $e^2 - e(e-1) = e$ and since $1, C, C^2, \dots, C^{e-1}$ belong to the center and are lin. ind., $1, C, C^2, \dots, C^{e-1}$ is a basis for the center.

(b) $\Delta(C) = \Delta(E)\Delta(F) + \frac{1}{q-q^{-1}}(q^{-1}\Delta(K) + q\,\Delta(K)^{-1})$

$= (E\otimes K + 1\otimes E)(K^{-1}\otimes F + F\otimes 1) + \frac{1}{q-q^{-1}}(q^{-1}K\otimes K + qK^{-1}\otimes K^{-1})$

$\in \operatorname{span}(E^aF^bK^c \otimes E^{a'}F^{b'}K^{c'} : a+b+a'+b' \le 2)$

Therefore $\Delta(C^n) \in \operatorname{span}(E^aF^bK^c\otimes E^{a'}F^{b'}K^{c'} : a+b+a'+b' \le 2n)$.

Now if $x = \beta_n C^n + \beta_{n-1}C^{n-1} + \dots + \beta_1 C + \beta_0 1$ is grouplike and $\beta_n \neq 0$, then

$$\Delta(x) = x\otimes x = \beta_n^2\, E^nF^n\otimes E^nF^n + B_n$$

where $B_n \in \operatorname{span}(E^aF^aK^c\otimes E^{a'}F^{a'}K^{c} : a+a' \le 2n-1)$

Hence $4n \le 2n$, i.e., $n=0$ and $x = 1$.

Ex 4

(a) Note first that $\mu \circ (\mathcal{S} \otimes id)(F^k K^j \otimes E^k K^i)$
$= \mathcal{S}(K)^j \mathcal{S}(F)^k E^k K^i$ and $\mathcal{S}(K) = K^{-1}$ and
$K\mathcal{S}(F)K^{-1} = \mathcal{S}(K^{-1})\mathcal{S}(F)\mathcal{S}(K) = \mathcal{S}(KFK^{-1}) = q^{-2}\mathcal{S}(F)$.

$$\Rightarrow \quad K\left(K^{-j}\mathcal{S}(F)^k E^k K^i\right) = K^{-j}\mathcal{S}(F)^k E^k K^i$$

$$\Rightarrow \quad KRK^{-1} = R$$

(b) It is enough to prove $\mathcal{S}(\mathcal{S}(x)) = KxK^{-1}$ for the generators. From $\mathcal{S}(K) = K^{-1}$ and

$$\begin{cases} \Delta(E) = E \otimes K + 1 \otimes E \\ \Delta(F) = K^{-1} \otimes F + F \otimes 1 \end{cases} \Longrightarrow \begin{cases} \mathcal{S}(E) = -EK^{-1} \\ \mathcal{S}(F) = -KF \end{cases}$$

Therefore

$$\begin{cases} \mathcal{S}(\mathcal{S}(K)) = \mathcal{S}(K^{-1}) = K = KKK^{-1} \\ \mathcal{S}(\mathcal{S}(E)) = \mathcal{S}(-EK^{-1}) = KEK^{-1} \\ \mathcal{S}(\mathcal{S}(F)) = \mathcal{S}(-KF) = KFK^{-1} \end{cases}.$$

Now $uxu^{-1} = KxK^{-1}$ for any $x \in A$.

$$\Rightarrow \quad K^{-1}u\, x = x K^{-1}u$$

(c) $K^{-2}u\,\mathcal{S}(u^{-1}) = K^{-1}u\, K^{-1}\mathcal{S}(u^{-1}) = K^{-1}u\,\mathcal{S}((K^{-1}u)^{-1})$
is clearly central. And by Ex 2 (d) it is grouplike because it is a product of grouplike elements. By Ex 3 the only grouplike central element is 1.

$$\Rightarrow \quad K^{-2}u\,\mathcal{S}(u^{-1}) = 1 \quad \Rightarrow \quad K^{-1}u = K\mathcal{S}(u) = \mathcal{S}(K^{-1}u)$$

(d) Let $\theta = K^{-1}u$. We already know that $\mathcal{S}(\theta) = \theta$. Also
$\varepsilon(\theta) = \varepsilon(K^{-1})\varepsilon(u) = 1$ and

$$\Delta(\theta) = \Delta(K^{-1})\Delta(u) = (K^{-1} \otimes K^{-1})(R_{21}R)^{-1}(u \otimes u)$$

commute by Ex 2 (b)

$$= (R_{21}R)^{-1}(\theta \otimes \theta)$$