Problem sheet 8

Exercise 1: *Two ways of writing the Yang-Baxter equation*
Let V be a vector space, and let R and $\check{R}$ be linear maps $V \otimes V \to V \otimes V$ related by

$$\check{R} \; = \; S_{V,V} \circ R.$$

Finish the proof discussed in the lecture, that the following equalities of linear maps $V \otimes V \otimes V \to V \otimes V \otimes V$ are equivalent

(i) $\check{R}_{12} \circ \check{R}_{23} \circ \check{R}_{12} \; = \; \check{R}_{23} \circ \check{R}_{12} \circ \check{R}_{23}$

(ii) $R_{12} \circ R_{13} \circ R_{23} \; = \; R_{23} \circ R_{13} \circ R_{12}$.

Exercise 2: *A solution to the Yang-Baxter equation*
Let V be a vector space with basis $v_1, v_2, \ldots, v_d$ and let $q \in \mathbb{C} \setminus \{0\}$. Define a linear map

$$\check{R} : V \otimes V \to V \otimes V \qquad \text{by} \qquad \check{R}(v_i \otimes v_j) \; = \; \begin{cases} q \, v_i \otimes v_j & \text{if } i = j \\ v_j \otimes v_i & \text{if } i < j \\ v_j \otimes v_i + \left(q - q^{-1}\right) v_i \otimes v_j & \text{if } i > j \end{cases} .$$

(a) Verify that we have the following equality of linear maps $V \otimes V \otimes V \to V \otimes V \otimes V$

$$\check{R}_{12} \circ \check{R}_{23} \circ \check{R}_{12} \; = \; \check{R}_{23} \circ \check{R}_{12} \circ \check{R}_{23}. \tag{YBE}$$

(b) Verify also that

$$(\check{R} - q \, \mathrm{id}_{V \otimes V}) \circ (\check{R} + q^{-1} \, \mathrm{id}_{V \otimes V}) \; = \; 0.$$

Exercise 3: *A universal R-matrix for* $\mathbb{C}[\mathbb{Z}/N\mathbb{Z}]$
Let N be a positive integer and denote $\omega = e^{2\pi \mathrm{i}/N}$. Let A be the algebra with one generator θ and relation $\theta^N = 1$, equipped with the unique Hopf algebra structure such that $\Delta(\theta) = \theta \otimes \theta$ (thus A is isomorphic to the Hopf algebra of the cyclic group $\mathbb{Z}/N\mathbb{Z}$). Define

$$R \; = \; \frac{1}{N} \sum_{m,n=0}^{N-1} \omega^{mn} \, \theta^m \otimes \theta^n \; \in \; A \otimes A.$$

Show that R is a universal R-matrix for A.

Exercise 4: *Properties of the universal R-matrix of braided Hopf algebras*
Let $A = (A, \mu, \Delta, \eta, \epsilon)$ be a braided bialgebra with universal R-matrix $R \in A \otimes A$.

(a) Show that $R_{21}^{-1} := S_{A,A}(R^{-1})$ is also a universal R-matrix for the bialgebra A.

Suppose furthermore that A admits an antipode $\gamma : A \to A$, so that $A = (A, \mu, \Delta, \eta, \epsilon, \gamma)$ is a braided Hopf algebra.

(b) Show that

$$(\gamma \otimes \mathrm{id}_A)(R) \; = \; R^{-1}.$$

(c) Show, for example using the results of (a) and (b), that

$$(\gamma \otimes \gamma)(R) \; = \; R.$$