Problem sheet 12

Recall:

- When q is a root of unity, we denote by e the smallest positive integer for which $q^e \in \{\pm 1\}$.
- For $\varepsilon \in \{\pm 1\}$ and $d \in \mathbb{N}$ we have an irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$-module W_d^ε with basis $w_0, w_1, \ldots, w_{d-1}$ defined by the formulas

$$\begin{aligned} K.w_j &= \varepsilon\, q^{d-1-2j}\, w_j \\ F.w_j &= w_{j+1} \\ E.w_j &= \varepsilon\, [j]_q\, [d-j]_q\, w_{j-1} \end{aligned}$$

  when either q is not a root of unity or when $d < e$
  (see lectures and *Problem sheet 11: Exercise 2*).
- When q is a root of unity, the indecomposable $\mathcal{U}_q(\mathfrak{sl}_2)$-module $W_e(\mu; a, b)$ is defined by the formulas

$$\begin{aligned} K.w_j &= \mu\, q^{-2j}\, w_j && \text{for } 0 \le j \le e-1 \\ F.w_j &= w_{j+1} && \text{for } 0 \le j \le e-2 \\ F.w_{e-1} &= b\, w_0 \\ E.w_j &= \Big(ab + \frac{[j]_q}{q-q^{-1}}(\mu q^{1-j} - \mu^{-1} q^{j-1})\Big)\, w_{j-1} && \text{for } 1 \le j \le e-1 \\ E.w_0 &= a\, w_{e-1} \end{aligned}$$

  (see *Problem sheet 11: Exercise 4*).

Exercise 1: *A family of indecomposable $\mathcal{U}_q(\mathfrak{sl}_2)$-modules of dimension e when q is a root of unity*
Let $q \in \mathbb{C}$ be a root of unity. Assume that $e > 1$.

(a) Show that $W_e(\mu; a, b)$ is irreducible unless $b = 0$ and $\mu \in \{\pm 1, \pm q, \pm q^2, \ldots, \pm q^{e-2}\}$.

(b) Consider the Hopf algebra $\widetilde{\mathcal{U}}_q(\mathfrak{sl}_2)$ which is the quotient of $\mathcal{U}_q(\mathfrak{sl}_2)$ by the ideal generated by E^e, F^e and $K^e - 1$ (cf. *Problem sheet 10: Exercise 3*). A $\widetilde{\mathcal{U}}_q(\mathfrak{sl}_2)$-module can be thought of as a $\mathcal{U}_q(\mathfrak{sl}_2)$-module, where E^e, F^e and $K^e - 1$ act as zero. Show that a $\widetilde{\mathcal{U}}_q(\mathfrak{sl}_2)$-module V is irreducible if and only if it is irreducible as a $\mathcal{U}_q(\mathfrak{sl}_2)$-module.

(c) Consider the modules W_d^ε for $d < e$, and the modules $W_e(\mu; a, b)$. Find all values of d and ε, and of μ, a, b for which these are irreducible $\widetilde{\mathcal{U}}_q(\mathfrak{sl}_2)$-modules in each of the following cases:

  - when e is odd and $q^e = +1$
    (*Answer:* d anything, $\varepsilon = +1$; $a = 0$, $b = 0$, $\mu = q^{-1}$; in fact $W_e(q^{-1}; 0, 0) \cong W_e^{+1}$)
  - when e is odd and $q^e = -1$
    (*Answer:* d anything, $\varepsilon = (-1)^{d-1}$; $a = 0$, $b = 0$, $\mu = -q^{-1}$; in fact $W_e(-q^{-1}; 0, 0) \cong W_e^{+1}$)
  - when e is even
    (*Answer:* d odd, ε anything; no possible values of μ, a, b)

Exercise 2: *A solution of Yang-Baxter equation from two-dimensional $\widetilde{\mathcal{U}}_q(\mathfrak{sl}_2)$-modules*
Let $q \in \mathbb{C}$ be a root of unity. Assume that $e > 1$, that e is odd and that $q^e = +1$. The finite dimensional algebra $\widetilde{\mathcal{U}}_q(\mathfrak{sl}_2)$ is generated by E, F, K with relations

$$KE = q^2\, EK \qquad KF = q^{-2}\, FK \qquad EF - FE = \frac{1}{q-q^{-1}}\left(K - K^{-1}\right)$$
$$E^e = 0 \qquad F^e = 0 \qquad K^e = 1.$$

It can be shown that the Hopf algebra $\widetilde{\mathcal{U}}_q(\mathfrak{sl}_2)$ is braided with the universal R-matrix

$$R = \frac{1}{e}\sum_{i,j,k=0}^{e-1} \frac{(q-q^{-1})^k}{[k]_q!} q^{k(k-1)/2+2k(i-j)-2ij}\, E^k K^i \otimes F^k K^j.$$

Let V be the two dimensional $\widetilde{\mathcal{U}}_q(\mathfrak{sl}_2)$-module W_2^{+1} with basis w_0, w_1 and calculate the matrix of

$$\check{R} = S_{V,V} \circ (\rho_V \otimes \rho_V)(R)$$

in the basis $w_0 \otimes w_0$, $w_0 \otimes w_1$, $w_1 \otimes w_0$, $w_1 \otimes w_1$.
Hint: In the calculations one encounters expressions of type $\sum_{t=0}^{e-1} q^{ts}$, for $s \in \mathbb{Z}$, which can be simplified significantly when q is a root of unity of order e.

Exercise 3: *Some tensor products of $\mathcal{U}_q(\mathfrak{sl}_2)$-modules*
Assume that $q \in \mathbb{C}$ is not a root of unity.

(a) Show that $W_d^\varepsilon \cong W_d^{+1} \otimes W_1^\varepsilon$.

(b) Let $d_1 \geq d_2 > 0$ and denote by $w_0^{(1)}, w_1^{(1)}, \ldots, w_{d_1-1}^{(1)}$ and $w_0^{(2)}, w_1^{(2)}, \ldots, w_{d_2-1}^{(2)}$ the bases of $W_{d_1}^{+1}$ and $W_{d_2}^{+1}$, respectively, chosen as in the beginning of the problem sheet. Consider the module $W_{d_1}^{+1} \otimes W_{d_2}^{+1}$. Show that for any $l \in \{0, 1, 2, \ldots, d_2 - 1\}$ the vector

$$v = \sum_{s=0}^{l} \frac{(-1)^s}{[s]_q!} \frac{[l]_q!}{[l-s]_q!} \frac{[d_1-1-s]_q!}{[d_1-1]_q!} \frac{[d_2-l-1+s]_q!}{[d_2-l-1]_q!}\, q^{s(2l-d_2-s)}\, w_s^{(1)} \otimes w_{l-s}^{(2)}$$

is an eigenvector of K and that it satisfies

$$E.v = 0.$$

(c) Using the result of (b), conclude that we have the following isomorphism of $\mathcal{U}_q(\mathfrak{sl}_2)$-modules

$$W_{d_1}^{+1} \otimes W_{d_2}^{+1} \cong W_{d_1+d_2-1}^{+1} \oplus W_{d_1+d_2-3}^{+1} \oplus W_{d_1+d_2-5}^{+1} \oplus \cdots \oplus W_{d_1-d_2+3}^{+1} \oplus W_{d_1-d_2+1}^{+1}.$$