Problem sheet 11

Exercise 1: *A lemma in Drinfeld double*
Let $A = (A, \mu, \Delta, \eta, \epsilon, \gamma)$ be a Hopf algebra with invertible antipode, and let A° be its restricted dual Hopf algebra. Below we denote the Drinfeld double associated to A and A° by $\mathcal{D}$, and the restricted dual Hopf algebra of it by $\mathcal{D}^\circ$.

Note that if $\phi \in \mathcal{D}^\circ$, then since the Hopf algebra A embeds to $\mathcal{D}$ by $\iota_A : A \to \mathcal{D}$, $a \mapsto a \otimes 1^*$, we can define an element $\phi|_A \in A^\circ$ by the formula

$$\langle \phi|_A, a \rangle \; = \; \langle \phi, \iota_A(a) \rangle$$

for all $a \in A$. Furthermore, since the Hopf algebra A° embeds to $\mathcal{D}$ by $\iota_{A^\circ} : A^\circ \to \mathcal{D}$, $\varphi \mapsto 1 \otimes \varphi$, we obtain an element $\hat{\phi} \in \mathcal{D}$ by setting

$$\hat{\phi} \; = \; \iota_{A^\circ}(\phi|_A).$$

Show that for all $\phi \in \mathcal{D}^\circ$ and $x \in \mathcal{D}$, the following identity holds in $\mathcal{D}$

$$\sum_{(\phi),(x)} \left\langle \phi_{(1)}, x_{(2)} \right\rangle x_{(1)} \hat{\phi}_{(2)} \; = \; \sum_{(\phi),(x)} \left\langle \phi_{(2)}, x_{(1)} \right\rangle \hat{\phi}_{(1)} x_{(2)}.$$

Exercise 2: *Irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$-modules of low dimension when q is a root of unity*
Let $q \in \mathbb{C}$ be a root of unity and denote by e the smallest positive integer such that $q^e \in \{+1, -1\}$. Assume $e > 1$. Consider the Hopf algebra $\mathcal{U}_q(\mathfrak{sl}_2)$.

(a) For $d < e$ a positive integer and $\varepsilon \in \{\pm 1\}$, show that the formulas

$$\begin{aligned} K.w_j &= \varepsilon \, q^{d-1-2j} \, w_j \\ F.w_j &= w_{j+1} \\ E.w_j &= \varepsilon \, [j]_q \, [d-j]_q \, w_{j-1} \end{aligned}$$

still define an irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$-module W_d^ε with basis $w_0, w_1, w_2, \ldots, w_{d-1}$.

(b) Show that any irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$-module of dimension less than e is isomorphic to a module of the above type.

Hint: Especially for (b), you can just check that the proofs we used when q was not a root of unity work in this case, too. Probably it is not worth going through this in great detail in the exercise session.

Exercise 3: *No irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$-modules of high dimension when q is a root of unity*
Let $q \in \mathbb{C}$ be a root of unity and denote by e the smallest positive integer such that $q^e \in \{+1, -1\}$. Assume $e > 1$. Suppose that one would have an irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$-module V with $\dim V > e$.

(a) If there exists a non-zero eigenvector $v \in V$ of K such that $F.v = 0$, then show that the linear span of $v, E.v, E^2.v, \ldots, E^{e-1}.v$ is a submodule of V. Conclude that this is not possible if V is irreducible and $\dim V > e$.

(b) If there doesn't exist any non-zero eigenvector $v \in V$ of K such that $F.v = 0$, then considering any non-zero eigenvector $v \in V$ of K, show that the linear span of $v, F.v, F^2.v, \ldots, F^{e-1}.v$ is a submodule of V. Conclude that this, too, is impossible if V is irreducible and $\dim V > e$.

(c) Conclude that there are no irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$-modules of dimension greater than e.

Hint for all parts of the exercise: Recall that K^e, E^e, F^e are central by *Problem Sheet 10: Exercise 3*, and remember also the central element $C = EF + \frac{1}{(q-q^{-1})^2}(q^{-1}K + qK^{-1})$.

Exercise 4: *A family of indecomposable $\mathcal{U}_q(\mathfrak{sl}_2)$-modules of dimension e when q is a root of unity*
Let $q \in \mathbb{C}$ be a root of unity and denote by e the smallest positive integer such that $q^e \in \{+1, -1\}$. Assume $e > 1$. Consider the Hopf algebra $\mathcal{U}_q(\mathfrak{sl}_2)$.

(a) Let $\mu, a, b \in \mathbb{C}$ with $\mu \neq 0$. Show that the following formulas define an e-dimensional $\mathcal{U}_q(\mathfrak{sl}_2)$-module with basis $w_0, w_1, w_2, \ldots, w_{e-1}$:

$$\begin{aligned} K.w_j &= \mu\, q^{-2j}\, w_j && \text{for } 0 \leq j \leq e-1 \\ F.w_j &= w_{j+1} && \text{for } 0 \leq j \leq e-2 \\ F.w_{e-1} &= b\, w_0 && \\ E.w_j &= \Big(ab + \frac{[j]_q}{q-q^{-1}}(\mu q^{1-j} - \mu^{-1} q^{j-1})\Big)\, w_{j-1} && \text{for } 1 \leq j \leq e-1 \\ E.w_0 &= a\, w_{e-1} && \end{aligned}$$

Denote this module by $W_e(\mu; a, b)$.

(b) Show that $W_e(\mu; a, b)$ is indecomposable, that is, it can not be written as a direct sum of two non-zero submodules.

(c) Show that $W_e(\mu; a, b)$ is irreducible unless $b = 0$ and $\mu \in \{\pm 1, \pm q, \pm q^2, \ldots, \pm q^{e-2}\}$.

(d) Consider the Hopf algebra $\widetilde{\mathcal{U}}_q(\mathfrak{sl}_2)$ which is the quotient of $\mathcal{U}_q(\mathfrak{sl}_2)$ by the ideal generated by E^e, F^e and $K^e - 1$ (cf. *Problem sheet 10: Exercise 3*). A $\widetilde{\mathcal{U}}_q(\mathfrak{sl}_2)$-module can be thought of as a $\mathcal{U}_q(\mathfrak{sl}_2)$-module, where E^e, F^e and $K^e - 1$ act as zero. Show that a $\widetilde{\mathcal{U}}_q(\mathfrak{sl}_2)$-module V is irreducible if and only if it is irreducible as a $\mathcal{U}_q(\mathfrak{sl}_2)$-module.

(e) Consider the modules W_d^ε of *Exercise 2*, for $d < e$, and the modules $W_e(\mu; a, b)$. Find all values of d and ε, and of μ, a, b for which these are irreducible $\widetilde{\mathcal{U}}_q(\mathfrak{sl}_2)$-modules in each of the following cases:

- when e is odd and $q^e = +1$
  (*Answer:* d anything, $\varepsilon = +1$; $a = 0$, $b = 0$, $\mu = q^{-1}$; in fact $W_e(q^{-1}; 0, 0) \cong W_e^{+1}$)
- when e is odd and $q^e = -1$
  (*Answer:* d anything, $\varepsilon = (-1)^{d-1}$; $a = 0$, $b = 0$, $\mu = -q^{-1}$; in fact $W_e(-q^{-1}; 0, 0) \cong W_e^{+1}$)
- when e is even
  (*Answer:* d odd, ε anything; no possible values of μ, a, b)