Dependence logic

Problems 5
Tuesday 3.5.2011

1. Let $\phi \in \Sigma_1^1(L \cup \{R\})$, where ϕ is in negation normal form and R is a k-ary relation symbol. We say that ϕ is downwards monotone with respect to R, if for all $L \cup \{R\}$ structures $(\mathcal{M}, A)$ (where $A \subseteq M^k$ interprets R) and $B \subseteq A$:

$$(\mathcal{M}, A) \models \phi \Rightarrow (\mathcal{M}, B) \models \phi.$$

Furthermore, we say that R appears in ϕ only negatively, if all subformulas of ϕ involving R are of the form $\neg R(t_1, ..., t_k)$. Show using induction on ϕ that, if R appears only negatively in ϕ, then ϕ is downwards monotone with respect to ϕ.

2. Let ϕ be as in the first exercise. Show that if ϕ is downwards monotone with respect to R, then there is a Σ_1^1-formula ϕ^* logically equivalent to ϕ in which R appears only negatively. (Hint: try replacing occurrences of R in ϕ by a new relation symbol.)

3. The connective called intuitionistic implication $\phi \twoheadrightarrow \psi$ is defined by

$$\mathcal{M} \models_X \phi \twoheadrightarrow \psi \text{ iff (for all } Y \subseteq X: \text{ if } \mathcal{M} \models_Y \phi \text{ then } \mathcal{M} \models_Y \psi).$$

Let $\mathcal{D}(\twoheadrightarrow)$ be the extension of dependence logic in which $\twoheadrightarrow$ is introduced as a new connective but negation is only allowed in front of atomic formulas ($\wedge$ and $\forall$ are also available). Show that dependence atoms $=\!(t_1, ..., t_k)$ can be expressed in $\mathcal{D}(\twoheadrightarrow)$ using only dependence atoms of the form $=\!(t_i)$.

4. Second-order logic is the extension of FO by universal and existential quantification of relation and function symbols. The interpretation of a formula of the form $\forall R\psi$, where R is k-ary, is that

$$\mathcal{M} \models_s \forall R\psi \Leftrightarrow (\mathcal{M}, S) \models_s \psi \text{ for all } S \subseteq M^k,$$

where S interprets R. Show that every formula $\phi \in \mathcal{D}(\twoheadrightarrow)$ can be translated to second-order logic. It suffices to extend the translation $\phi \mapsto \tau_{1,\phi}$ between dependence logic and Σ_1^1 by a clause corresponding to $\twoheadrightarrow$. (see Theorem 6.2 of the course textbook on page 88.)

5. Let ϕ be a sentence of $\mathcal{D}(\twoheadrightarrow)$. Construct a sentence $\psi \in \mathcal{D}(\twoheadrightarrow)$ such that for all $\mathcal{M}$:

$$\mathcal{M} \models \psi \Leftrightarrow \mathcal{M} \not\models \phi.$$

This shows that, for sentences, $\mathcal{D}(\twoheadrightarrow)$ is closed under classical negation.

6. Every sentence $\phi \in \mathcal{D}$ is logically equivalent to a sentence $\psi \in \mathcal{D}$ of the form

$$\forall x_1 \ldots \forall x_n \exists x_1 \ldots \exists x_m (\theta_1 \wedge \theta_2), \tag{1}$$

where θ_1 is a conjunction of dependence atoms and θ_2 is a quantifier-free first-order formula. Is the following generalization of this result possible: Every sentence $\phi \in \mathcal{D}$ is strongly logically equivalent to a sentence as in (1)?