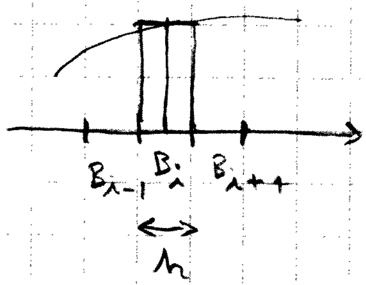
Computational Statistics
Second exam 13. 5. 2011

1) We try to find an interval (a, b) large enough so that

$$\int_{\mathbb{R}} k(\theta)p(\theta)p(y|\theta)\,d\theta \approx \int_a^b k(\theta)p(\theta)p(y|\theta)\,d\theta$$

$$\int_{\mathbb{R}} p(\theta)p(y|\theta)\,d\theta \approx \int_a^b p(\theta)p(y|\theta)\,d\theta$$

We divide the interval (a,b) into N pieces of equal width h, and apply the midpoint rule on each of the subintervals:



$$\int_{B_i} g(\theta)\,d\theta \approx h\, g(\theta_i)$$

θ_i = midpoint of subinterval B_i

We get

$$E[k(\Theta)\,|\,Y=y] = \int_{\mathbb{R}} k(\theta)\,p(\theta|y)\,d\theta$$

$$= \frac{\int_{\mathbb{R}} k(\theta)p(\theta)p(y|\theta)\,d\theta}{\int_{\mathbb{R}} p(\theta)p(y|\theta)\,d\theta}$$

$$\approx \frac{\int_a^b k(\theta)p(\theta)p(y|\theta)\,d\theta}{\int_a^b p(\theta)p(y|\theta)\,d\theta}$$

$$\approx \frac{\cancel{h}\sum_{i=1}^N k(\theta_i)p(\theta_i)p(y|\theta_i)}{\cancel{h}\sum_{j=1}^N p(\theta_j)p(y|\theta_j)}$$

[Here $B_i = (a_i, b_i)$ $\quad h = (b-a)/N$

$a_i = a + h(i-1) \quad b_i = a + hi$

$\theta_i = a + h(i - \frac{1}{2}), \quad i=1,\ldots,N$]

2) The proposal θ' is calculated as follows

$$v \sim \mathrm{Exp}(1)$$
$$\theta' = \theta v$$

The proposal density is

$$q(\theta'|\theta) = \mathrm{Exp}(v|1) \left|\frac{dv}{d\theta'}\right| = e^{-\theta'/\theta} \frac{1}{\theta} \; ; \; \theta, \theta' > 0$$

where $\theta' = \theta v \iff v = \theta'/\theta$

Alternatively, one can recognize that the proposal distribution is $\mathrm{Exp}(1/\theta)$ [we parametrize Exp by the rate parameter] and

$$q(\theta'|\theta) = \mathrm{Exp}(\theta' | \tfrac{1}{\theta}) = \frac{1}{\theta} e^{-\theta'/\theta}$$

MH sampler:

Let $\theta_0 > 0$ be the initial state
for $i = 0, \ldots, N-1$
    draw $v \sim \mathrm{Exp}(1)$
    $\theta' \leftarrow v \theta_i$
    draw $u \sim \mathrm{Uni}(0,1)$

$$r \leftarrow \frac{p(\theta')\, p(y|\theta')\, \exp(-\theta_i/\theta')\, 1/\theta'}{p(\theta_i)\, p(y|\theta_i)\, \exp(-\theta'/\theta_i)\, 1/\theta_i}$$

    if $(u < r)$ then
        $\theta_{i+1} \leftarrow \theta'$
    else
        $\theta_{i+1} \leftarrow \theta_i$
    endif
endfor

3) $p(y, \varphi, \gamma, \tau) = p(y \mid \varphi, \gamma, \tau)\, p(\tau)\, p(\varphi)\, p(\gamma)$

$[\varphi, \gamma, \tau]$

$$\propto \prod_{i=1}^{\tau} \binom{k}{y_i} \varphi^{y_i} (1-\varphi)^{k-y_i} \prod_{i=\tau+1}^{n} \binom{k}{y_i} \gamma^{y_i} (1-\gamma)^{k-y_i}$$

$$\varphi^{a_1-1}(1-\varphi)^{b_1-1} \gamma^{a_2-1} (1-\gamma)^{b_2-1}$$

$$\propto \varphi^{a_1 + \sum_1^\tau y_i - 1} (1-\varphi)^{b_1 + \sum_1^\tau (k-y_i) - 1}$$

$$\gamma^{a_2 + \sum_{\tau+1}^n y_i - 1} (1-\gamma)^{b_2 + \sum_{\tau+1}^n (k-y_i) - 1}$$

Here $0 < \varphi < 1$, $0 < \gamma < 1$, $1 \le \tau \le n-1$.

We see:

$$[\varphi \mid \gamma, \tau, y] \sim Be\left(a_1 + \sum_{i=1}^{\tau} y_i,\ b_1 + \sum_{i=1}^{\tau} (k-y_i)\right)$$

$$[\gamma \mid \varphi, \tau, y] \sim Be\left(a_2 + \sum_{i=\tau+1}^{n} y_i,\ b_2 + \sum_{i=\tau+1}^{n} (k-y_i)\right)$$

$[\tau \mid \varphi, \gamma, y]$ is a discrete distribution on $1 \le \tau \le n-1$ with pmf

$$p(\tau \mid \varphi, \gamma, y) = \frac{p(y, \varphi, \gamma, \tau)}{\sum_{\tau'=1}^{n-1} p(y, \varphi, \gamma, \tau')}$$

One step of Gibbs sampler:

Draw

$$\varphi_{i+1} \sim Be\left(a_1 + \sum_{j=1}^{\tau_i} y_j,\ b_1 + \sum_{j=1}^{\tau_i} (k-y_j)\right)$$

$$\gamma_{i+1} \sim Be\left(a_2 + \sum_{j=\tau_i+1}^{n} y_j,\ b_2 + \sum_{j=\tau_i+1}^{n} (k-y_j)\right)$$

τ_{i+1} from the discrete distribution with pmf $p(\tau \mid \varphi_{i+1}, \gamma_{i+1}, y)$ on $1 \le \tau \le n-1$

[It is critical to condition on the most recently updated values of the other parameters in the pseudocode.]

4 a) Search the maximum of posterior density, or equivalently, search minimum of

$$\theta \mapsto -\log[\, p(\theta)\, p(y|\theta)\,]$$

$$b := \arg\min\left[-\log p(\theta)\, p(y|\theta) \right]$$

$Q :=$ Hessian of $-\log[p(\theta)p(y|\theta)]$ calculated at $\hat{\theta} = b$

b) $$p(y) = \int p(\theta)\, p(y|\theta)\, d\theta$$

$$= \int \exp\{ \log[p(\theta)\, p(y|\theta)] \}\, d\theta$$

$$\approx \int \exp\{ \log[p(\tilde{\theta})p(y|\tilde{\theta})] + 0 - \tfrac{1}{2}(\theta-\tilde{\theta})^T Q(\theta-\tilde{\theta}) \}\, d\theta$$

[∵ by second order Taylor expansion at $\theta = \tilde{\theta} := b$]

$$= \log[p(\tilde{\theta})p(y|\tilde{\theta})] \, \frac{(2\pi)^{k/2}}{\sqrt{\det Q}}$$

[∵ by calculating the integral based on the normalizing constant of the multivariate normal]

c) Generate $\theta_1, \dots, \theta_N$ from $t_k(\nu, b, Q^{-1})$
Let g be the density of $t_k(\nu, b, Q^{-1})$

$$p(y) = \int p(\theta)\, p(y|\theta)\, d\theta = \int \frac{p(\theta)p(y|\theta)}{g(\theta)}\, g(\theta)\, d\theta$$

$$\approx \frac{1}{N} \sum_{i=1}^{N} \frac{p(\theta_i)\, p(y|\theta_i)}{g(\theta_i)}$$

[Importance sampling estimator for marginal likelihood $p(y)$]