1 a) The prior is improper since

$$\int_0^\infty \frac{1}{\theta}\, d\theta \left(= \Big|_0^\infty \ln\theta \right) = \infty$$

and therefore the function $\frac{1}{\theta}$ cannot be normalized to become a pdf.

b) $$p(\theta|y) \propto p(\theta)\, p(y|\theta)$$
$$\propto \frac{1}{\theta} \sqrt{\theta}\, e^{-\frac{1}{2}\theta y^2}$$
$$\propto \theta^{\frac{1}{2}-1} e^{-\frac{1}{2}y^2\theta} \propto \mathrm{Gam}(\theta \mid \tfrac{1}{2}, \tfrac{1}{2}y^2)$$

Therefore the posterior is $\mathrm{Gam}(\frac{1}{2}, \frac{1}{2}y^2)$ if this is a proper distribution. It is proper if both of its parameters are positive, and this is the case, if $y \neq 0$.

2 a) $$p(y|\theta) = \prod_{i=1}^n p(y_i|\theta) = \prod_{i=1}^n \theta e^{-\theta y_i}$$
$$= \theta^n \exp\left(-\left(\sum_{i=1}^n y_i\right)\theta\right)$$

b) $\phi = 1/\theta \iff \theta = 1/\phi$

Likelihood in terms of the new parameter is

$$p(y|\phi) = \theta^n \exp\left(-\left(\sum_{i=1}^n y_i\right)\theta\right)$$
$$= \phi^{-n} \exp\left(-\left(\sum_{i=1}^n y_i\right)\frac{1}{\phi}\right)$$

c) Prior for the new parameter is

$$p(\phi) = \pi(\theta) \left|\frac{d\theta}{d\phi}\right| = \pi\left(\frac{1}{\phi}\right)\frac{1}{\phi^2}, \quad \phi > 0$$

[When we make a change of variables on the right-hand side of $p(\cdot|\cdot)$, we don't need the Jacobian; on the left-hand side we need it!]

3) It seems hopeless to calculate the cdf and then the quantile function, so simulation by inversion seems impossible (or would require hard work). On the other hand, $f^*(x)$ is easy to bound with something familiar:

since $0 \leq \sin^2 x \leq 1 \quad \forall x > 0 \Rightarrow \frac{1}{2+\sin^2 x} \leq \frac{1}{2+0} \quad \forall x > 0$

therefore

$$f^*(x) \leq \frac{1}{2} e^{-x}, \text{ when } x > 0,$$

where e^{-x} is the density of $\mathcal{E}xp(1)$ (for $x > 0$).

For $x \leq 0$ we have trivially

$$0 = f^*(x) \leq \tfrac{1}{2} g(x) = 0$$

where $g(x)$ is the density of $\mathcal{E}xp(1)$.

Hence we can use accept-reject with $\mathcal{E}xp(1)$ as the instrumental distribution:

```
repeat
    generate x ~ Exp(1) and u ~ Uni(0,1).
until ( 1/2 u e^(-x) < e^(-x) / (2 + sin^2 x) )
return(x)
```

When we repeat this simulation many times, we get the desired simulation.

4) a) Write

$$I = \int_0^\infty \underbrace{\frac{x}{2+\sin^2 x}}_{h(x)} \underbrace{e^{-x}}_{f_0(x) \text{ (density of } Exp(1))} dx$$

$$\hat{I} = \frac{1}{N}\sum_{i=1}^{N} h(x_i) = \frac{1}{N}\sum_{i=1}^{N} \frac{x_i}{2+\sin^2 x_i},$$

where $x_1, \dots, x_N$ is a sample drawn from $Exp(1)$.

b) $\mathrm{var}\,\hat{I} = \frac{1}{N}\mathrm{var}\left(\frac{X}{2+\sin^2(X)}\right)$, where $X \sim Exp(1)$

We get the Monte Carlo standard error by plugging in the sample estimate of the variance and by taking the square root:

$$\hat{se}(\hat{I}) = \sqrt{\frac{1}{N}}\sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(\frac{x_i}{2+\sin^2 x_i} - \hat{I}\right)^2}$$

c)

$$EX = \int x f(x)\,dx = \frac{\int x f^*(x)\,dx}{\int f^*(x)\,dx}$$

$$= \frac{\int x \frac{f^*(x)}{g(x)} g(x)\,dx}{\int \frac{f^*(x)}{g(x)} g(x)\,dx}$$

I select g to be the density of $Exp(1)$:

1) Draw $x_1, \dots, x_N$ from $Exp(1)$

2) Calculate the importance weights

$$w_i = \frac{f^*(x_i)}{g(x_i)} = \frac{1}{2+\sin^2(x_i)}, \quad i = 1, \dots, N$$

3) Estimate $\mu = EX$ by

$$\hat{\mu} = \frac{\frac{1}{N}\sum_{i=1}^{N} w_i x_i}{\frac{1}{N}\sum_{j=1}^{N} w_j}$$