1

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κ is strongly inaccessible. Show:

a.) $(\mathcal{P}(x))^{V_\kappa} = \mathcal{P}(x)$

This means: "the object a that V_κ thinks is $\mathcal{P}(x)$ in V_κ is in fact $\mathcal{P}(x)$."

So let $V_\kappa \models a = \mathcal{P}(x)$. $\overset{!}{\Rightarrow}$ $a = \mathcal{P}(x)$

$$y \in a \underset{\substack{V_\kappa \text{ is} \\ \text{transitive}}}{\Rightarrow} y \in V_\kappa \, . \underset{\text{by assumption}}{(y \subseteq x)^{V_\kappa}} \Rightarrow \underset{\text{absoluteness}}{y \subseteq x} \Rightarrow y \in \mathcal{P}(x).$$

$$\in \mathcal{P}(x) \Rightarrow y \subseteq x \in V_\alpha \Rightarrow y \in V_{\alpha+1} \subseteq V_\kappa \Rightarrow (y \subseteq x)^{V_\kappa} \Rightarrow$$

κ is limit

$(y \in a)^{V_\kappa} \Rightarrow$

$y \in a$ (absoluteness

26.) $(\omega_\alpha)^{V_\kappa} = \omega_\alpha$.

By induction on α.

Case $\alpha = 0$: $\omega_0 = \omega$ and ω is absolute.

Case $\alpha+1$: Want $(\omega_{\alpha+1})^{V_\kappa} = (\omega_\alpha)^+ = \omega_{\alpha+1}$.

Now $\omega_\alpha \underset{\text{ind hyp}}{=} (\omega_\alpha)^{V_\kappa} < (\omega_{\alpha+1})^{V_\kappa}$, so $(\omega_{\alpha+1})^{V_\kappa} > \omega_\alpha$.

Now sp. $\beta < (\omega_{\alpha+1})^{V_\kappa} \Rightarrow |\beta| \leq \omega_\alpha$.

Now $V_\kappa \models \beta < \omega_{\alpha+1}$ so $\exists f \in V_\kappa$: $V_\kappa \models f : \beta \xrightarrow{1-1} \omega_\alpha$.

"f is a 1-1 function" is absolute.

So $f : \beta \xrightarrow{1-1} (\omega_\alpha)^{V_\kappa} \underset{\text{ind hyp.}}{=} \omega_\alpha$ So $|\beta| \leq \omega_\alpha$.

Case α = limit ordinal : want $(\omega_\alpha)^{V_\kappa} = \omega_\alpha$.

Let $\beta \in \omega_\alpha$. Then since α is limit, $\beta \in \omega_\rho$, $\rho < \alpha$.

Then $\beta \underset{\text{ind hyp}}{\in} (\omega_\rho)^{V_\kappa}$. So $\omega_\alpha \subseteq (\omega_\alpha)^{V_\kappa}$.

Now let $\beta \in (\omega_\alpha)^{V_\kappa}$. Then $V_\kappa \models \beta \in \omega_\alpha$. Now ω_α is lim in V_κ ("$x = \rho+1$" is absolute), so $V_\kappa \models \beta \in \omega_\rho$, $\rho < \alpha$.

Ie $\beta \in (\omega_\rho)^{V_\kappa} = \omega_\rho$. So $\beta \in \omega_\alpha$.

c.) κ s.i. Then

$(\beth_\alpha)^{V_\kappa} = \beth_\alpha$. (Ie $(y = \beth_\alpha)^{V_\kappa}$ and $y = \beth_\alpha$)

By ind on α:

Case 1 $\alpha = 0$. Then $\beth_0 = \omega$, which is absolute

Case 2 $\alpha = \beta+1$ Now $(\beth_\alpha)^{V_\kappa} = (2^{\beth_\beta})^{V_\kappa}$

We would like to know that $\underbrace{(2^{\beth_\beta})^{V_\kappa} = 2^{\beth_\beta}}_{(*)}$ $(= \beth_\alpha$

Pf of (*):

$(2^{\beth_\beta})^{V_\kappa} = |\mathcal{P}(\beth_\beta)|^{V_\kappa}$. Let $\alpha = |\mathcal{P}(\beth_\beta)|^{V_\kappa}$ Then

$\exists \pi \in V_\kappa$ st $\left[\pi : \alpha \xrightarrow{1-1} \mathcal{P}(\beth_\beta)^{V_\kappa}\right]^{V_\kappa}$

We claim that $\alpha = |\mathcal{P}(\beth_\beta)|$.

Now $V \models \pi : \alpha \xrightarrow{1-1} (\mathcal{P}(\beth_\beta))^{V_\kappa} \iff$

$|\alpha| = |\mathcal{P}(\beth_\beta)^{V_\kappa}| \iff$ by absoluteness of $\mathcal{P}(x)$ ie $\mathcal{P}$

$|\alpha| = |\mathcal{P}(\beth_\beta^{V_\kappa})| \iff$

$|\alpha| = |\mathcal{P}(\beth_\beta)|$ by ind hyp

Now we show α is a cardinal: otherwise $\exists$

$\sigma : \alpha \xrightarrow{1-1} \beta$, $\beta < \alpha$. $\alpha \in V_\kappa$, $\beta \in V_\kappa \Rightarrow \sigma \in V_\kappa$.

$\sigma \subseteq (\alpha \times \beta)$ and this is downwards absolute. Hence $V_\kappa \models$

Hence α is cardinal. $\square$

c, cont:

<u>case 3</u> α a limit. then $(\beth_\alpha)^{V_\kappa} = (\sup\{\beth_\gamma \mid \gamma<\alpha\})^{V_\kappa}$

<u>Claim.</u> $(\beth_\alpha)^{V_\kappa} = \beth_\alpha$. taking sups is absolute! this is Δ_0.

why? $$(\beth_\alpha)^{V_\kappa} = \Big(\sup_{\beta<\alpha} \beth_\beta\Big)^{V_\kappa} = \sup_{\beta<\alpha} (\beth_\beta)^{V_\kappa}$$

$$= \underbrace{\sup_{\beta<\alpha} \beth_\beta}_{\text{by ind hyp}} = \beth_\alpha.$$

Recall:
$\sup A = \bigcup A$.

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d.) $(V_\alpha)^{V_\kappa} = V_\alpha$.

Case $\alpha = \beta + 1$:

$$(V_\alpha)^{V_\kappa} = (\mathcal{P}(V_\beta))^{V_\kappa} \underbrace{=}_{\text{by part a}} \mathcal{P}((V_\beta)^{V_\kappa}) \underbrace{=}_{\text{ind hy}} \mathcal{P}(V_\beta) = V_\alpha.$$

$\alpha \in$ limit.

$$\left(\bigcup_{\gamma<\alpha} V_\gamma\right)^{V_\kappa} \underbrace{=}_{\text{"}\bigcup\text{" is absolute}} \bigcup_{\gamma<\alpha} (V_\gamma)^{V_\kappa} \underbrace{=}_{\text{ind hyp}} \bigcup_{\gamma<\alpha} V_\gamma = V_\alpha$$

e.) See p. 6

f.) $(\alpha \text{ is s.i.})^{V_\kappa} \leftrightarrow \alpha$ is s.i.

~~Pf~~ $\Rightarrow$: Sp. α is singular in V. Then $\exists$ cofinal map j, $j: \gamma \to \alpha$ in V, $\gamma < \alpha$. $\gamma \in V_\kappa$, $\therefore j \in V_\kappa$ (κ s.i.), ↯.

$\Leftarrow$: Sp. $V_\kappa \models \omega_\beta < \alpha$. Then $V_\kappa \models 2^{\omega_\beta} < \alpha$, as o/w the negation is upwards persistent.

(Note that α must be regular in V_κ, o/w a cofinal map persists upwards.)

(5.5) 2f continued.

We have shown that α $\in$ regular $\longleftrightarrow$ $(\alpha \in \text{regular})^{V_\kappa}$

Now must show that $(\lambda < \kappa \to 2^\lambda < \kappa)^V$

$\longleftrightarrow$

$(\lambda < \kappa \to 2^\lambda < \kappa)^{V_\kappa}$

$\Leftarrow$: So sp α is s i in V_κ and sp $\lambda < \alpha$ and $\alpha \leq 2^\lambda$ $\quad 2^\lambda = |\mathcal{P}(\lambda)|$ so $\exists f : \alpha \xrightarrow{1-1} (\mathcal{P}(\lambda))^V$.

Now $f \subseteq \alpha \times \mathcal{P}(\lambda)$ $\quad \alpha \in V_\beta$, some β, $\lambda \in V_\rho$,

So $\mathcal{P}(\lambda) \in V_\sigma$ $\therefore$ $f \in V_\eta$, some η ie

$f \in V_\kappa$.

$\therefore$ $V_\kappa \models$ "$f : \alpha \xrightarrow{1-1} (\mathcal{P}(\lambda))^V$" But $(\mathcal{P}(\lambda))^V = (\mathcal{P}(\lambda))^{V_\kappa}$,

$\not\!\!\downarrow$.

$\Rightarrow$: Sp. $(\alpha \text{ s.i.})^V$ Claim $(\alpha \in \text{s.i})^{V_\kappa}$.

Sp. $V_\kappa \models \lambda < \alpha$. Claim. $V_\kappa \models 2^\lambda < \alpha$.

Otherwise $V_\kappa \models \exists f : \alpha \xrightarrow{1-1} (\mathcal{P}(\lambda))^{V_\kappa}$

But this again is $(\mathcal{P}(\lambda))^V$.

$\therefore$ $(\exists f : \alpha \xrightarrow{1-1} \mathcal{P}(\lambda))^V$, $\not\!\!\downarrow$.