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Laver, Richard (1-CO)
Certain very large cardinals are not created in small forcing extensions. (English summary)
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A. Lévy and R. M. Solovay [Israel J. Math. **5** (1967), 234–248; MR0224458 (37 #57)] showed that if κ is a cardinal, P is a partial order of cardinality less than κ, and $G \subseteq P$ is a V-generic filter, then κ is a measurable cardinal in V if and only if it is a measurable cardinal in $V[G]$. J. D. Hamkins and W. H. Woodin [Proc. Amer. Math. Soc. **128** (2000), no. 10, 3025–3029; MR1664390 (2000m:03121)] proved the analogous result for strong cardinals and Woodin cardinals. In this paper, the author proves the analogous result for a family of very strong large cardinal axioms. For an ordinal n in the interval $[0, \omega]$, and cardinals $\kappa < \lambda$, the author lets $E_n(\kappa, \lambda)$ denote the statement that there exists an elementary embedding from V_λ to V_λ with critical point κ which is moreover Σ^1_n-elementary for subsets of V_λ. Kunen's argument [K. Kunen, J. Symbolic Logic **36** (1971), 407–413; MR0311478 (47 #40)] for the nonexistence of an elementary embedding from V to itself implies that λ must be a strong limit cardinal of cofinality ω in this context. For $n \in \omega$, Martin [cf. R. Laver, Ann. Pure Appl. Logic **90** (1997), no. 1-3, 79–90; MR1489305 (99c:03074)] proved that $E_{2n+1}(\kappa, \lambda)$ implies $E_{2n+2}(\kappa, \lambda)$. The main theorem of the paper is the following : If n is an ordinal in the interval $[0, \omega]$, κ is a cardinal, P is a partial order of cardinality less than κ, $G \subseteq P$ is a V-generic filter, and $V[G] \models \exists\lambda E_n(\kappa, \lambda)$, then $V \models \exists\lambda E_n(\kappa, \lambda)$. The author notes that the opposite (upwards) direction is proved in the same manner as for smaller cardinals. He also shows that if k is an embedding witnessing $E_n(\kappa, \lambda)$ in such an extension $V[G]$, then it follows that $k \restriction V_\alpha \in V$ for all $\alpha < \lambda$, but it need not hold that $k \restriction V_\lambda \in V$.

On the way to proving his main theorem, the author uses work of J. D. Hamkins [Fund. Math. **180** (2003), no. 3, 257–277; MR2063629 (2005m:03100)] to prove the following interesting fact: If V is a model of ZFC, P is a partial order, $\delta = (|P|^+)^V$ and $G \subseteq P$ is a V-generic filter, then V is definable in $V[G]$ from the parameter $V_{\delta+1}$.

In his concluding remarks, the author notes that the upwards direction of his result holds for the stronger large cardinal hypothesis (due to Woodin) that there exists an elementary embedding of the form $j\colon V_{\lambda+1} \to V_{\lambda+1}$ which extends to an elementary embedding from $L[V_{\lambda+1}]$ to itself. The downward direction is left open, though the author notes that Woodin has proved it under the assumption that $(V_{\lambda+1})^\#$ exists.

Reviewed by *Paul Bradley Larson*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.